CS320: Problems and Solutions for Days 9–13, Winter 2023

Problem 1 For each of the following six languages, give the *best possible* classification of that language into one of the following classes:

regular
context-free
recursive
recursively enumerable
not recursively enumerable

Your classification of a language is the best possible if the class which you indicate indeed contains that language, while no other listed class which is a proper subset of your class contains that language.

1. The language consisting of pairs $(R(M), w)$, where $R(M)$ is a representation of Turing machine M and M halts on input w.

   Answer: Recursively enumerable.

2. The language consisting of representations of Turing machines that do not halt on blank tape.

   Answer: Not recursively enumerable.

3. $\emptyset$.

   Answer: Regular.

4. $\{0, 1, 111, 00\}$.

   Answer: Regular.

5. The language generated by the grammar:
   $G = \{\{S\}, \{a, b\}, P, S\}$, where P is given by:
   $S \to aS \mid bS \mid \lambda$.

   Answer: Regular.

6. The union of two arbitrary recursive languages.

   Answer: Recursive.

Problem 2 **(a)** Give an example of a language that is not regular but has an infinite regular subset and an infinite regular superset. Give a precise definition of these three languages and explain your answer briefly. If such a language does not exist, explain why.

Answer: One example is the complement $\overline{L_2}$ of the language L_2, defined as follows:

$$L_2 = \{a^k b^k \mid k \geq 0\}$$

L_2 is a canonical non-regular language, as is readily proved by Pumping Lemma. Hence, its complement $\overline{L_2}$ is not regular. However, $\overline{L_2}$ contains infinite regular subsets, like $\boldsymbol{a}^*$ or $\boldsymbol{b}^*$. Straightforwardly, $\overline{L_2}$ is a subset of $(\boldsymbol{a} \cup \boldsymbol{b})^*$, which is evidently regular, and thereby a regular superset of $\overline{L_2}$.

(b) Give an example of a language that is not context-free but has an infinite context-free subset and an infinite context-free superset. Give a precise definition of all these three languages and explain your answer briefly. If such a language does not exist, explain why.

Answer: The language:

$$L_4 = \{a^k b^m c^k d^m \mid k, m \geq 0\}$$

is not context-free, as is readily proved by Pumping Lemma. However, its subset:

$$L_2 = \{a^k c^k \mid k \geq 0\}$$

(obtained by fixing $m = 0$ in the template for L_4) is a canonical context-free language. Straightforwardly, L_4 is a subset of $\{a, b, c, d\}^*$, which is evidently regular and thereby context-free.

(c) Give an example of a language that is not recursively enumerable but has a recursively enumerable complement. Give a precise definition of this language and explain your answer briefly. If such a language does not exist, explain why.

Answer: One example is the complement $\overline{L_H}$ of the halting-problem language, defined as follows:

$$L_H = \{(M, w) \mid (M, w) \searrow\}$$

L_H is the set of all pairs (machine,input) such that the machine halts on the input. L_H is recursively enumerable—the universal Turing machine accepts it. However, since L_H is known not to be decidable (recursive), its complement $\overline{L_H}$ cannot be recursively enumerable.

Problem 3 For each of the following claims, circle the word "**yes**" that follows the claim if the claim is correct, and circle the word "**no**" that follows the claim if the claim is not correct.

1. The union of any two regular languages is context-free.

   Answer: Yes, in fact it is regular.

2. The intersection of any two regular languages is regular.

   Answer: Yes.

3. The union of any two context-free languages is regular.

   Answer: No—but it is context-free.

4. The intersection of any two context-free languages is context-free.

   Answer: No—some context-free languages do not have a context-free intersection.

5. Every subset of a regular language is regular.

   Answer: No—all languages (including non-regular ones) are subsets of Σ^*, which is certainly regular.

6. Every regular language is context-free.

   Answer: Yes.

7. Some context-free languages are not regular.

   Answer: Yes.

8. Some context-free languages are regular.

   Answer: Yes.

9. Every non-deterministic finite automaton has an equivalent deterministic finite automaton.

   Answer: Yes—the construction is algorithmic.

10. Some non-deterministic push-down automata have equivalent regular expressions.

    Answer: Yes—some context-free languages are regular.

11. Some finite automata have equivalent regular grammars.

    Answer: Yes—in fact all of them do.

12. Every finite automaton has an equivalent regular grammar.

    Answer: Yes.

13. Every regular grammar has an equivalent regular expression.

    Answer: Yes—the construction is algorithmic.

14. Every context-free grammar has an equivalent non-deterministic finite automaton.

    Answer: No—some context-free languages are not regular.

15. Every regular expression has an equivalent deterministic finite automaton.

    Answer: Yes–the construction is algorithmic.

16. Every non-deterministic push-down automaton has an equivalent regular grammar.

    Answer: No—some context-free languages are not regular.

17. Every subset of a context-free language is context-free.

    Answer: No—all languages (including those that are not context-free) are subsets of Σ^*, which is certainly regular, and thereby context-free.

18. The intersection of any regular language and any context-free language is context-free.

    Answer: Yes.

19. The intersection of any regular language and any context-free language is regular.

    Answer: No—but it is context-free.

20. The union of any regular language and any context-free language is regular.

    Answer: No—but it is context-free.

21. Every context-free language contains a regular subset.

    Answer: Yes—all languages contain $\emptyset$, which is certainly regular.

22. The complement of every context-free language is context-free.

    Answer: No—some context-free languages do not have a context-free complement.

23. The complement of every regular language is context-free.

    Answer: Yes—in fact it is regular.

24. The complement of every regular language is regular.

    Answer: Yes.

25. Every recursively enumerable language is accepted by final state by some Turing machine.

    Answer: Yes—by definition.

26. Some recursively enumerable languages are not accepted by halting by any Turing machine.

    Answer: No—by definition.

27. Every recursive language is recursively enumerable.

    Answer: Yes—by definition.