CS320: Problems and Solutions for Days 1–8, Winter 2023

Problem 1 Let:

$$\Sigma = \{a_1, a_2, \ldots, a_k\}$$
$$V = \{A_1, A_2, \ldots, A_m\}$$

and let:

$$N = \{0, 1, \ldots\}$$

be the set of natural numbers. State the cardinality of each of the following sets. (For a finite set, state the exact number. For an infinite set, specify if it is countable or not.)

Answer:

1. $\Sigma \times V$ has km elements.
2. V^* is infinite and countable.
3. $(\Sigma \times V)^*$ is infinite and countable.
4. $\mathcal{P}(\Sigma)$ has 2^k elements.
5. $\mathcal{P}(V^*)$ is infinite and uncountable.
6. set of total functions from V to Σ kas k^m elements.
7. set of total functions from N to V is infinite and uncountable.
8. set of all context-free grammars of the form (V, Σ, P, S) is infinite and countable.
9. set of all regular expressions over Σ is infinite and countable.
10. language defined by the regular expression $\boldsymbol{(a_1 \cup a_2)(a_2 a_3)^*}$ is infinite and countable.
11. $N \times \Sigma$ is infinite and countable.
12. set of all languages over Σ is infinite and uncountable.

Problem 2 Let:

$$\Sigma = \{a, b, c, d, e\}$$
$$V = \{A, B, D, E, F, H, J, K\}$$

State the cardinality of each of the following sets. (For a finite set, state the exact number. For an infinite set, specify if it is countable or not.)

Answer:

1. $\Sigma \cup V$ has $5 + 8 = 13$ elements.
2. $\Sigma^* \cup V^*$ is infinite and countable.
3. $\Sigma \cap V = \emptyset$, hence it has 0 elements.
4. $(\Sigma \cap V)^* = \emptyset^* = \{\lambda\}$, hence it has 1 element.
5. $\mathcal{P}(V)$ has $2^8 = 256$ elements.
6. $\mathcal{P}(\Sigma^*)$ is infinite and uncountable.
7. set of all regular expressions over Σ is infinite and countable.
8. set of all languages over Σ is infinite and uncountable.
9. set of total functions from Σ to V has 8^5 elements.
10. set of all context-free grammars of the form (V, Σ, P, S) is infinite and countable.
11. $V \times \Sigma$ has $8 \times 5 = 40$ elements.

12. set of all languages over V is infinite and uncountable.

Problem 3 For each of the following claims, circle the word "**yes**" that follows the claim if the claim is correct, and circle the word "**no**" that follows the claim if the claim is not correct.

1. The class of context-free languages is closed under concatenation. $\boxed{\textbf{yes}}$
2. The Kleene star of any regular language is regular. $\boxed{\textbf{yes}}$
3. The Kleene star of any regular language is context-free. $\boxed{\textbf{yes}}$
4. The union of any two context-free languages is context-free. $\boxed{\textbf{yes}}$
5. Every finite language is regular. $\boxed{\textbf{yes}}$
6. There exists an algorithm to convert any regular expression into an equivalent finite automaton. $\boxed{\textbf{yes}}$
7. There exists an algorithm to convert any regular expression into an equivalent context-free grammar. $\boxed{\textbf{yes}}$
8. Every deterministic finite automaton is equivalent to some non-deterministic finite automaton. $\boxed{\textbf{yes}}$
9. Every non-deterministic finite automaton is equivalent to some deterministic finite automaton. $\boxed{\textbf{yes}}$
10. Every context-free grammar is equivalent to some regular expression. $\boxed{\textbf{no}}$
11. Set of strings of the form $\{a^k b^k c^k \mid k \geq 0\}$ is regular. $\boxed{\textbf{no}}$
12. Set of strings of the form $\{a^k b^k c^k \mid k \geq 0\}$ is context-free. $\boxed{\textbf{no}}$
13. Set of strings of the form $\{a^k b^k \mid k \geq 0\}$ is regular. $\boxed{\textbf{no}}$
14. Set of strings of the form $\{a^k b^k \mid k \geq 0\}$ is context-free. $\boxed{\textbf{yes}}$
15. Every language has a finite description. $\boxed{\textbf{no}}$
16. Every finite language has a finite description. $\boxed{\textbf{yes}}$
17. Set $\{a, b\}^*$ is regular. $\boxed{\textbf{yes}}$
18. Every subset of $\{a, b\}^*$ is regular. $\boxed{\textbf{no}}$
19. Every subset of $\{a, b\}^*$ is context-free. $\boxed{\textbf{no}}$

Problem 4 For each of the following claims, circle the word "**yes**" that follows the claim if the claim is correct, and circle the word "**no**" that follows the claim if the claim is not correct.

1. The concatenation of any two regular languages is context-free. $\boxed{\textbf{yes}}$
2. The Kleene star of any regular language is context-free. $\boxed{\textbf{yes}}$
3. Every subset of a regular language is regular. $\boxed{\textbf{no}}$
4. Some context-free languages are not regular. $\boxed{\textbf{yes}}$
5. Some finite languages are not regular. $\boxed{\textbf{no}}$
6. Some regular languages are not finite. $\boxed{\textbf{yes}}$
7. The union of any two regular languages is regular. $\boxed{\textbf{yes}}$
8. Language $\{a^n b^m c^k \mid k = m + n,\ k, m, n \geq 0\}$ is context-free. $\boxed{\textbf{yes}}$
9. Language $\{a^n b^m c^k \mid k, m, n \geq 0\}$ is not regular. $\boxed{\textbf{no}}$

10. Language $\{a^n b^m c^k d^\ell \mid n = k \wedge m = \ell,\ k, m, n, \ell \geq 0\}$ is context-free. $\boxed{\textbf{no}}$

11. Language $\{a^n b^m c^k d^\ell \mid n = k \vee m = \ell,\ k, m, n, \ell \geq 0\}$ is context-free. $\boxed{\textbf{yes}}$

12. Every regular language has a context-free subset. $\boxed{\textbf{yes}}$

13. Every infinite context-free language has an infinite regular superset. $\boxed{\textbf{yes}}$

14. Language $\{a^n b^m c^k d^\ell \mid n = k \wedge m = \ell,\ k, m, n, \ell \geq 0\}$ has a proper subset which is context-free and infinite. $\boxed{\textbf{yes}}$

15. Every infinite language has a finite description. $\boxed{\textbf{no}}$

16. Every finite language is generated by some context-free grammar. $\boxed{\textbf{yes}}$

17. Some infinite languages cannot be described by a regular expression. $\boxed{\textbf{yes}}$

18. Set $\{a, b\}^*$ has a proper subset which is infinite and not context-free. $\boxed{\textbf{yes}}$

19. The intersection of $\{a^n b^m c^m \mid m, n \geq 0\}$ and $\{a^n b^n c^m \mid m, n \geq 0\}$ is context-free. $\boxed{\textbf{no}}$

20. The intersection of $\{a^n b^m c^m \mid m, n \geq 0\}$ and $\{a^n b^n c^m \mid m, n \geq 0\}$ has a context-free complement. $\boxed{\textbf{yes}}$

Problem 5 Fill the empty box at the end of each of the numbered claims with one of these signs:

$$-,\ \surd,\ ?$$

so that the meaning of the signs is as follows:

- $-$ means that the preceding claim is always false;
- $\surd$ means that the preceding claim is always true;
- ? means that the preceding claim may be true but may be false, depending on the values of the variable(s) appearing in the claim.

Problem assumptions: L_1 is an arbitrary regular language; L_2 is an arbitrary context-free language; e_1 is a regular expression that represents L_1; G_2 is a context-free grammar that generates L_2.

Claims:

1. L_1 is context-free. $\boxed{\surd}$
2. L_2 is regular. $\boxed{?}$
3. L_2^* is context-free. $\boxed{\surd}$
4. L_1 has a regular complement. $\boxed{\surd}$
5. $L_1 \cap L_2$ is regular. $\boxed{?}$
6. $L_1 \cap L_2$ is context-free. $\boxed{\surd}$
7. L_2 has a context-free complement. $\boxed{?}$
8. $L_1 L_2$ is context-free. $\boxed{\surd}$
9. There exists an algorithm which on input e_1 outputs a regular context-free grammar for L_1. $\boxed{\surd}$
10. There exists an algorithm which on input G_2 outputs a regular context-free grammar for L_2. $\boxed{-}$