CS320: Problems and Solutions for Day 3, Winter 2023

Problem 1 Let $N = \{0, 1, \ldots\}$ be the set of natural numbers and let L be the language defined by the regular expression:

$$(\boldsymbol{a} \cup \boldsymbol{\lambda})\,(\boldsymbol{a} \cup \boldsymbol{\lambda})\,\boldsymbol{b}^*$$

Construct a bijection from L to N. If such bijection does not exist, prove it.

Answer: Observe that L is a union of 3 sets:

$$\boldsymbol{b}^* \;\cup\; \boldsymbol{a}\,\boldsymbol{b}^* \;\cup\; \boldsymbol{aa}\,\boldsymbol{b}^*$$

which means that every string in L is of the form:

$$a^r b^q\ ,\ q \geq 0,\ r \in \{0, 1, 2\}$$

Set $\{0, 1, 2\}$ is the set of all possible remainders in the division by 3. We map $a^r b^q$ to that number that gives remainder r and quotient q when divided by 3:

$$f : a^r b^q \to 3q + r$$

To see that f is surjective, note that every natural number has a quotient and a remainder in the division by 3, and thereby has an original in L under f. To see that f is injective, observe that the representation by the pair (quotient, remainder) is unique. Two distinct strings from L differ in the number of a's or in the number of b's. Hence, their images differ in their quotients or in their remainders in the division by 3, which suffices to guarantee that they are different.

Problem 2 Let:

$$\Sigma = \{0, 1\}$$

and let L be the language defined by the regular expression:

$$(\mathbf{0} \cup \mathbf{1})\,\mathbf{01}$$

Let N be the set of natural numbers.

State the cardinality of each of the following sets. (For a finite set, state the exact number. For an infinite set, specify if it is countable or not.)

Answer:

1. $|L| = 2$, since $L = \{001, 101\}$.
2. $|L^*| = \aleph_0$, i.e., L^* is infinite and countable.
3. $|L^0| = 1$, since $L^0 = \{\lambda\}$.
4. set of all strings of length 5 over Σ has 2^5 elements.
5. set of all strings of finite length over Σ is infinite and countable.
6. set of subsets of Σ has $2^2 = 4$ elements.
7. set of subsets of Σ^* is infinite and uncountable.
8. set of all languages over Σ is infinite and uncountable.
9. set of all functions from N to Σ is infinite and uncountable.
10. set of regular expressions over Σ is infinite and countable.

Problem 3 **(a)** Give an example of a finite language that is not regular. Give a precise definition of this language and explain your answer briefly. If such a language does not exist, explain why.

Answer: Impossible, every finite language is regular.

(b) Give an example of a regular language that is not finite. Give a precise definition of this language and explain your answer briefly. If such a language does not exist, explain why.

Answer: $\boldsymbol{a}^*$ is evidently regular because it has a regular expression, and also infinite because it contains a string of any length.

(c) Give an example of an infinite uncountable language. Give a precise definition of this language and explain your answer briefly. If such a language does not exist, explain why.

Answer: Impossible—every language is countable, since it is a set of finite sequences drawn from a countable set.

Problem 4 Let L be the language defined by the regular expression:

$$(\boldsymbol{c} \cup \boldsymbol{ab} \cup \boldsymbol{b})^* \boldsymbol{bb}\,(\boldsymbol{c} \cup \boldsymbol{ab})$$

(a) Write 10 distinct strings that belong to L. If such strings do not exist, explain why.

(b) Write 10 distinct strings over alphabet $\{a, b, c\}$ that do not belong to L. If such strings do not exist, explain why.

Answer:

$\in L$	$\notin L$
bbc	λ
$bbab$	bb
$cbbc$	bba
$abbbc$	$bbca$
$bbbc$	bbb
$ccabbbbbc$	a
$bbcbbab$	b
$bbabbbab$	ab
$ccbcbbab$	$aabb$
$cabcbcbbab$	$abbab$

Problem 5 Let L be a set of strings over alphabet $\{0, 1\}$ such that

$$L = \{w \mid w \neq 11 \wedge w \neq 111\}$$

(a) Write 5 distinct strings that belong to L. If such strings do not exist, explain why.

Answer:

$$\lambda, 0, 1, 00, 01, 10, 000, 001, 010, 011, 100, 101, 110$$

(b) Write 5 distinct strings over alphabet $\{0, 1\}$ that do not belong to L. If such strings do not exist, explain why.

Answer: It is impossible to find 5 such strings. Straightforwardly, by definition of L, there are only two strings over $\{0, 1\}$ that do not belong to L:

$$11, 111$$

(c) Write a regular expression that defines L. If such expression does not exist, explain why.

Answer:

$$\boldsymbol{\lambda} \cup \mathbf{1} \cup (\mathbf{0} \cup \mathbf{10} \cup \mathbf{110} \cup \mathbf{111}(\mathbf{0} \cup \mathbf{1}))(\mathbf{0} \cup \mathbf{1})^*$$

Problem 6 Let L be the language defined by the regular expression:

$$(\boldsymbol{a} \cup \boldsymbol{b})(\boldsymbol{a} \cup \boldsymbol{b})(\boldsymbol{a} \cup \boldsymbol{b})^*$$

(a) Write 8 distinct strings that belong to L. If such strings do not exist, explain why.

Answer:

$$aa, ab, ba, bb, aab, abb, baa, bbb$$

(b) Write 8 distinct strings over alphabet $\{a, b\}$ that do not belong to L. If such strings do not exist, explain why.

Answer: By inspection of the regular expression, we conclude that L consists of all strings over $\{a, b\}$ that have length greater than or equal to 2. Hence, there are only three strings over $\{a, b\}$, precisely those with length less than 2:

$$\lambda, a, b$$

that are not in L. Thus, it is impossible to list more than 3 of them.

Problem 7 Let L_1 be the language defined by the regular expression:

$$(\boldsymbol{a} \cup \boldsymbol{b})\ (\ (\boldsymbol{a} \cup \boldsymbol{b})(\boldsymbol{a} \cup \boldsymbol{b})\ )^*$$

and let L_2 be the language defined by the regular expression:

$$(\boldsymbol{a} \cup \boldsymbol{b})\ (\boldsymbol{a} \cup \boldsymbol{b})\ (\boldsymbol{a} \cup \boldsymbol{b})^*$$

In your answers to the following questions, state cardinalities of finite sets by giving the exact numbers. For infinite sets, specify if they are countable or not.

(a) Let $S_1 = L_1 \cup L_2$. Write a regular expression that defines S_1. If such a regular expression does not exist, explain why.

Answer: Observe that L_1 is the set of all strings of odd length over $\{a, b\}$, while L_2 is the set of all strings of length greater than 1 over $\{a, b\}$. Their union S_1 is the set of all strings of length greater than zero:

$$(\boldsymbol{a} \cup \boldsymbol{b})\ (\boldsymbol{a} \cup \boldsymbol{b})^*$$

(b) Let $S_2 = L_1 \setminus L_2$. Write a regular expression that defines S_2. If such a regular expression does not exist, explain why.

Answer: S_2 is the set of strings of length equal to 1:

$$(\boldsymbol{a} \cup \boldsymbol{b})$$

(c) Let $S_3 = L_1 \cap L_2$. Write a regular expression that defines S_3. If such a regular expression does not exist, explain why.

Answer: S_3 is the set of strings of odd length, greater than 1:

$$(\boldsymbol{a} \cup \boldsymbol{b})\ (\boldsymbol{a} \cup \boldsymbol{b})\ (\boldsymbol{a} \cup \boldsymbol{b})\ (\ (\boldsymbol{a} \cup \boldsymbol{b})(\boldsymbol{a} \cup \boldsymbol{b})\ )^*$$

(d) What is the cardinality of S_1?

Answer: $|S_1| = \aleph_0$; S_1 is infinite and countable.

(e) What is the cardinality of S_2?

Answer: $|S_2| = 2$.

(f) What is the cardinality of S_3?

Answer: $|S_3| = \aleph_0$; S_3 is infinite and countable.

Problem 8 **(a)** Let L_1 be the set of all strings of length two or more over alphabet $\{a, b\}$ in which all the a's follow all the b's. Write 5 distinct strings that belong to L_1. If such strings do not exist, explain why.

Answer:

$$aa, bb, ba, bbb, baa$$

(b) Write a regular expression that defines L_1. If such a regular expression does not exist, explain why.

Answer:

$$\boldsymbol{bbb^*a^*} \cup \boldsymbol{baa^*} \cup \boldsymbol{aaa^*}$$

(c) Let L_2 be the set of all strings over alphabet $\{a, b, c\}$ that begin with c, end with b, and contain exactly two a's. Write 5 distinct strings that belong to L_2. If such strings do not exist, explain why.

Answer:

$$caab, ccaab, cabab, ccabab, cbababb$$

(d) Write a regular expression that defines L_2. If such a regular expression does not exist, explain why.

Answer:

$$\boldsymbol{c}\ (\boldsymbol{b} \cup \boldsymbol{c})^*\ \boldsymbol{a}\ (\boldsymbol{b} \cup \boldsymbol{c})^*\ \boldsymbol{a}\ (\boldsymbol{b} \cup \boldsymbol{c})^*\ \boldsymbol{b}$$

Problem 9 **(a)** Let L_1 be the set of all strings over alphabet $\{0, 1\}$ that do not contain the substring 11. Write a regular expression that defines L_1. If such regular expression does not exist, explain why.

Answer:

$$\mathbf{(0 \cup 10)^*(1 \cup \lambda)}$$

(b) Let L_2 be the set of all strings over alphabet $\{0, 1\}$ that contain an odd number of 1's.
Write a regular expression that defines L_2. If such regular expression does not exist, explain why.

Answer:

$$\mathbf{0^*10^*(0^*10^*10^*)^*}$$

Problem 10 **(a)** Let L_1 be the set of all strings over alphabet $\{0, 1\}$ that contain exactly two zeros.
Write a regular expression that defines L_1. If such regular expression does not exist, explain why.

Answer:

$$\mathbf{1^*01^*01^*}$$

(b) Let L_2 be the set of all strings over alphabet $\{0, 1\}$ that contain at least two zeros.
Write a regular expression that defines L_2. If such regular expression does not exist, explain why.

Answer:

$$\mathbf{(0 \cup 1)^*0(0 \cup 1)^*0(0 \cup 1)^*}$$

(c) Let L_3 be the set of all strings over alphabet $\{0, 1\}$ that contain at most two zeros.
Write a regular expression that defines L_3. If such regular expression does not exist, explain why.

Answer:

$$\mathbf{1^*(0 \cup \lambda)1^*(0 \cup \lambda)1^*}$$

Problem 11 **(a)** Let L be the set of strings over the alphabet $\{0, 1\}$ whose first symbol is different from the last symbol. Write a regular expression that defines L. If such a regular expression does not exist, prove it.

Answer:

$$\mathbf{0(0 \cup 1)^*1 \cup 1(0 \cup 1)^*0}$$

(b) Let $\mathcal{R}$ be the class of languages that can be represented by a regular expression, and let $\mathcal{N}$ be the class of languages that cannot be represented by a regular expression. State the cardinalities of $\mathcal{R}$ and $\mathcal{N}$, and compare them.

Answer: Class $\mathcal{R}$ is infinite and countable:

$$|\mathcal{R}| = \aleph_0$$

Class $\mathcal{N}$ is infinite and uncountable; its cardinality is equal to the cardinality of the set of subsets of an infinite countable set:

$$|\mathcal{N}| = 2^{\aleph_0}$$

Hence:

$$|\mathcal{N}| > |\mathcal{R}|$$

Problem 12 Let L be the set of all strings over alphabet $\{a, b, c\}$ whose first letter occurs at least once again in the string.

Write a regular expression that defines L. If such a regular expression does not exist, prove it.

Answer:

$$\boldsymbol{a(b \cup c)^*a(a \cup b \cup c)^*}$$
$$\bigcup$$
$$\boldsymbol{b(a \cup c)^*b(a \cup b \cup c)^*}$$
$$\bigcup$$
$$\boldsymbol{c(a \cup b)^*c(a \cup b \cup c)^*}$$