CS320: Problems and Solutions for Day 2, Winter 2023

Problem 1 Let N be the set of natural numbers and let $\mathcal{F}$ be the set of functions from the set N to the set $\{0,1\}$.
(a) Construct an injective function $g_1 : N \to \mathcal{F}$. If such a function does not exist, explain why.
Answer: Let the image of element $i \in N$ be the function $c_i \in \mathcal{F}$:

$$g_1(i) = c_i$$

where c_i is defined as follows:

$$c_i(x) = \begin{cases} 1 \text{ if } x = i \\ 0 \text{ if } x \neq i \end{cases}$$

(b) Construct an injective function $g_2 : \mathcal{F} \to N$. If such a function does not exist, explain why.
Answer: Since

$$|\mathcal{F}| = |\{0,1\}|^{|N|} = 2^{\aleph_0} > \aleph_0 = |N|$$

there does not exist an injection from set $\mathcal{F}$ to set N.

Problem 2 Let N be the set of natural numbers and let $\mathcal{F}$ be the set of functions from set N to set $\{0,1\}$.
(a) Construct an injective function:

$$g_1 : N \to \mathcal{F}$$

If such function does not exist, explain why.
Answer: Let the image of element $i \in N$ be the function $c_i \in \mathcal{F}$:

$$g_1(i) = c_i$$

where c_i is defined as follows:

$$c_i(x) = \begin{cases} 1 \text{ if } x \leq i \\ 0 \text{ if } x > i \end{cases}$$

(b) Construct a surjective function:

$$g_2 : N \to \mathcal{P}(N)$$

If such function does not exist, explain why.
Answer: Since

$$|\mathcal{P}(N)| > |N|$$

there does not exist a surjection from the set N to the set $\mathcal{P}(N)$.
(c) Construct a bijective function:

$$g_3 : \mathcal{P}(N) \to \mathcal{F}$$

If such function does not exist, explain why.
Answer: Let $g_3(X) = f_X$, where

$$f_X(i) = \begin{cases} 1 \text{ if } i \in X \\ 0 \text{ if } i \notin X \end{cases}$$

Problem 3 Let:

$$\Sigma = \{a, b, c\}$$

and let:

$$N = \{0, 1, \ldots\}$$

be the set of natural numbers. State the cardinality of each of the following sets. (For a finite set, state the exact number. For an infinite set, specify if it is countable or not.)
Answer:

1. $|\mathcal{P}(\Sigma)| = 2^3 = 8$
2. $|\Sigma^*| = \aleph_0$
   (countable)

3. $|$ set of total functions from Σ to $\{0,1\}$ $| = 2^3 = 8$
4. $|$ set of total functions from Σ^* to $\{0,1\}$ $| > \aleph_0$
(uncountable)
5. $|\mathcal{P}(\Sigma^*)| > \aleph_0$
(uncountable)
6. $|$ set of total functions from N to Σ $| > \aleph_0$
(uncountable)
7. $|\Sigma \times \Sigma| = 3 \times 3 = 9$
8. $|\Sigma^* \times \Sigma^*| = \aleph_0$
(countable)
9. $|$ set of all finite subsets of N $| = \aleph_0$
(countable)
10. $|\mathcal{P}(N)| > \aleph_0$
(uncountable)

Problem 4 **(a)** Give an example of two countably infinite sets, S_1 and S_2, such that $S_1 \cap S_2$ is finite. (Define S_1 and S_2 precisely.) If such sets do not exist, explain why.

Answer: There are infinitely many correct answers. A trivial one is to employ two disjoint countably infinite sets, say the set of positive natural numbers and the set of negative natural numbers. Their intersection is empty and thereby finite.

For a less trivial example, where S_1 and S_2 have a non-empty intersection, let:

$$\begin{array}{ccc} S_1 : \boldsymbol{a^*b} & S_2 : \boldsymbol{ab^*} & S_1 \cap S_2 : \boldsymbol{ab} \\ |S_1| = \aleph_0 & |S_2| = \aleph_0 & |S_1 \cap S_2| = 1 \end{array}$$

(b) Give an example of two countably infinite sets, S_1 and S_2, such that $S_1 \setminus S_2$ is infinite. (Define S_1 and S_2 precisely.) If such sets do not exist, explain why.

Answer: There are infinitely many correct answers. A trivial one is to employ two disjoint countably infinite sets, say the set of positive natural numbers and the set of negative natural numbers. Then: $S_1 \setminus S_2 = S_1$, which is infinite.

For a less trivial example, where S_1 and S_2 have an infinite intersection, let:

$$\begin{array}{ccc} S_1 : \boldsymbol{(a \cup b)^*} & S_2 : \boldsymbol{a^*} & S_1 \setminus S_2 : \boldsymbol{(a \cup b)^* b (a \cup b)^*} \\ |S_1| = \aleph_0 & |S_2| = \aleph_0 & |S_1 \setminus S_2| = \aleph_0 \end{array}$$

Problem 5 Let $N = \{0, 1, \ldots\}$ be the set of natural numbers. Construct five different injective functions from N to $N \times N$. If such functions do not exist, explain why.

Answer: There are infinitely many correct answers. For example:

$$\begin{aligned} f_1(x) &= (0, x) \\ f_2(x) &= (x, 1) \\ f_3(x) &= (17, 5x) \\ f_4(x) &= (2^x, 12) \\ f_5(x) &= (x^2, x^3) \end{aligned}$$

Problem 6 Let $N = \{0, 1, \ldots\}$ be the set of natural numbers.

(a) Construct an injective function from $N \times N$ to N. Justify your answer briefly. If such a function does not exist, explain why.

Answer: There are infinitely many correct answers. A simple one:

$$f : (x, y) \mapsto 2^x \cdot 3^y$$

To see that f is injective, we have to show that

$$f(x, y) = f(v, w)$$

is impossible unless $(x, y) = (v, w)$. Observe that 2 and 3 are prime. Hence:

$$2^x \cdot 3^y = 2^v \cdot 3^w$$

is possible only if:

$$x = v \text{ and } y = w$$

which means:

$$(x, y) = (v, w)$$

whence the claim.

(b) Construct a non-injective function from $N \times N$ to N. Justify your answer briefly. If such a function does not exist, explain why.

Answer: There are infinitely many correct answers. A trivial one:

$$f : (x, y) \mapsto 0$$

Since f sends the entire set $N \times N$ to a single element of N, f is trivially non-injective.

For a less trivial example, let:

$$g : (x, y) \mapsto x$$

g is surjective, and every element of N is the image of infinitely many elements of $N \times N$. Precisely, all pairs of the form:

$$(a, y), \text{ for } y = 0, 1, 2, \ldots$$

are mapped to a under f.

Problem 7 **(a)** Construct an injective function from $N \times N$ to $N \times N \times N$. Justify your answer briefly. If such a function does not exist, explain why.

Answer: There are infinitely many such functions. A simple solution is:

$$f : (x, y) \to (x, y, 0)$$

This function is injective since it is essentially the identity function in $N \times N$.

(b) Construct an injective function from $N \times N \times N$ to $N \times N$. Justify your answer briefly. If such a function does not exist, explain why.

Answer: There are infinitely many such functions. A simple solution is:

$$f : (x, y, z) \to (2^x \cdot 3^y \cdot 5^z, 0)$$

Observe that the second component of the target pair is always equal to 0. By the properties of the prime-factor decomposition of integers, two distinct triplets are mapped to two pairs that differ in their first component, whence the injectiveness.

(c) Construct a surjective function from $N \times N \times N$ to $N \times N$. Justify your answer briefly. If such a function does not exist, explain why.

Answer: There are infinitely many such functions. A simple solution is:

$$f : (x, y, z) \to (x, y)$$

This function is surjective since it is a projection of the identity function in $N \times N \times N$. Exactly $\aleph_0$ elements of $N \times N \times N$ are mapped into each element of $N \times N$, precisely all those triplets that agree in the first two components and differ in the third component.