CS320: Problems and Solutions for Day 11, Winter 2023

Problem 1 You are given two Turing machines, M_1 and M_2, such that M_1 *accepts* language L_1 and M_2 *decides* language L_2.

Is $L_1 \setminus L_2$ a recursively enumerable language?
If your answer is "yes", prove it by describing an appropriate Turing machine. If your answer is "no", prove it by showing that such a Turing machine does not exist.

Answer: Yes—$L_1 \setminus L_2$ is recursively enumerable. Let $L = L_1 \setminus L_2$. A Turing machine M that accepts L operates as follows. Given an input string w, M first simulates M_2 until M_2 halts, which must happen because M_2 *decides* language L_2. If M_2 accepts then M rejects, because $w \in L_2$ implies $w \notin L$. If M_2 rejects, then M simulates M_1, and halts and accepts if and when M_1 halts and accepts, which occurs if and only if $w \in L_1$. Since also $w \notin L_2$, M indeed accepts if and only if $w \in L_1 \setminus L_2$.

Problem 2 You are given two Turing machines, M_1 and M_2, such that M_1 *accepts* language L_1 and M_2 *accepts* language L_2.

Is $L_1 \cup L_2$ a recursively enumerable language?
If your answer is "yes", prove it by describing an appropriate Turing machine. If your answer is "no", prove it by showing why such a Turing machine does not exist.

Answer: Yes, $L_1 \cup L_2$ is a recursively enumerable language. A Turing machine M' that accepts it operates as follows. On a given input string w, M' emulates simultaneously the operation of M_1 and M_2. Say, M' emulates M_1 on its even steps, and M' emulates M_2 on its odd steps. M' halts and accepts w if and only if one of the machines M_1, M_2 halts and accepts w.

Problem 3 Let:

$$L = \{(R(M), n) \mid M \text{ halts on blank tape after } \leq n \text{ steps } \}$$

where $R(M)$ is a representation of Turing machine M and n is a natural number. Describe a Turing machine M' that accepts L. If such M' does not exist, explain why.

Answer: M' simulates M on a blank-tape input and counts the simulated steps. If M halts before n steps are counted, M' accepts.

Problem 4 Let L be a non-recursive language, accepted by a Turing machine M, and let k be a natural number. Describe a Turing machine M', such that on input w, M' writes *error* on its tape and halts if and only if M does not accept w within the first k computation steps. If such M' does not exist, explain why.

Answer: M' simulates M on input w, and counts the simulated steps. If and when the number of steps reaches k, M' does as follows. If M has not (yet) accepted w or if M has rejected w then M' writes *error* and halts. If M accepts w before the number of steps reaches k then M' diverges.