CS320: Problems for Day 1, Winter 2023

Problem 1 The sets A and B are defined as follows:

$$A = \{a, b, c, d, e\}$$
$$B = \{0, 1, 2\}$$

(a) How many functions are there from set A to the set B?

Answer: There are $|B|^{|A|} = 3^5 = 243$ functions from set A to set B.

(b) Construct a function $f_1 : A \to B$. If such a function does not exist, explain why.

Answer:

$$f_1(a) = f_1(b) = f_1(c) = f_1(d) = f_1(e) = 0$$

(c) Construct an injective function $f_2 : A \to B$. If such a function does not exist, explain why.

Answer: Since $|A| = 5 > 3 = |B|$, there does not exist an injection from set A to set B.

(d) Construct a surjective function $f_3 : A \to B$. If such a function does not exist, explain why.

Answer:

$$f_3(a) = f_3(b) = f_3(c) = 0, f_3(d) = 1, f_3(e) = 2$$

(e) How many partial functions are there from set A to the set B?

Answer: A partial function f can have 4 effects on an element $a \in A$. (Either $f(a)$ is one of the 3 elements in B, or it is undefined.) Therefore, there are $(|B| + 1)^{|A|} = 4^5 = 1024$ partial functions from set A to set B.

Problem 2 Let sets A and B be defined as follows:

$$A = \{a, b, c, d, e\}$$
$$B = \{0, 1, 2\}$$

and let N be the set of natural numbers.

(a) Construct an injective function f_1 from B to $A \times B$. If such a function does not exist, explain why.

Answer: Since $|A \times B| = 5 \times 3 = 15$, there are exactly $15 \times 14 \times 13 = 2730$ different injections from B to $A \times B$. Here is one of them:

$$f_1(x) = (a, x)$$

(b) Construct a surjective function f_2 from A to a proper subset of N. If such a function does not exist, explain why.

Answer: Choose set $\{6\} \subset N$ to be the range of f_2 and define f_2 as follows:

$$f_2(x) = 6$$

(c) Construct an injective function f_3 from A to $\mathcal{P}(B)$. If such a function does not exist, explain why.

Answer: Since $|\mathcal{P}(B)| = 2^3 = 8$, there are exactly $8 \times 7 \times 6 \times 5 \times 4 = 6720$ different injections from A to $\mathcal{P}(B)$. Here is one of them:

x	$f_3(x)$
a	$\emptyset$
b	$\{0\}$
c	$\{1\}$
d	$\{2\}$
e	$\{0, 1\}$

(d) Construct a proper subset S_1 of $\mathcal{P}(B)$ which is infinite. If such a set does not exist, explain why.

Answer: Impossible—$\mathcal{P}(B)$ is a finite set (with 8 elements) and cannot have an infinite subset.

(e) Construct a proper subset S_2 of N which is infinite. If such a set does not exist, explain why.

Answer: For example:

The set of even natural numbers:

$$S_2 = \{2x \mid x \in N\}$$