CS320: Problems for Day 4, Winter 2023

Problem 1 Let L be a language over alphabet $\{a, b\}$ with the following property:

For every word $w \in L$:
if $|w|$ is odd, then the middle symbol of w is a;
if $|w|$ is even, then w ends with bb.

Write a complete formal definition of a context-free grammar G that generates L. If such grammar G does not exist, explain why.

Problem 2 Construct a context-free grammar G over alphabet $\{a, b, c\}$ that generates the language

$$L(G) = \{a^m b^n c^i \mid m + n < i\}$$

where m, n, i are non-negative integers.

Problem 3 **(a)** Does there exist a pair of languages L_1 and L_2 such that all of the following three conditions hold?

- L_1 is regular;
- $L_2 \subseteq L_1$;
- L_2 is not regular, but is context-free.

If your answer to this part is "no" go to part **(d)**, else complete parts **(b)**–**(c)**.
(b) Write a regular expression that defines L_1 (as in part **(a)**).
(c) Write a complete formal definition of a context-free grammar that generates L_2 (as in part **(a)**). Describe L_2 briefly, using words and set-notation.
(d) Explain why such a pair of languages L_1 and L_2 (as in part **(a)**) does not exist.

Problem 4 Let L be the language generated by the context-free grammar $G = \{V, \Sigma, P, S\}$, where $\Sigma = \{a, b, c\}$, $V = \{S, A, B\}$, and the production set P is:

$$S \to aSb \mid A$$
$$A \to cAb \mid B$$
$$B \to cb$$

(a) Write 8 distinct strings that belong to L. If such strings do not exist, explain why.
(b) Write 8 distinct strings over alphabet $\{a, b, c\}$ that do not belong to L. If such strings do not exist, explain why.

Problem 5 Write a complete formal definition of a context-free grammar G that generates language L, defined as follows:

$$L = \{a^m b^n ac \mid m \geq 0, n > m\}$$

If such a grammar does not exist, explain why.

Problem 6 Let:

$$L = \{a^m b^m \mid m \geq 0\}$$

Write a complete formal definition of a context-free grammar that generates the complement of L in $\{a, b\}^*$. If such grammar does not exist, prove it.

Problem 7 Let L_1 be the language defined by the regular expression:

$$\boldsymbol{(a \cup b)^* \, c \, (a \cup b)^* \, c \, (a \cup b)^* \, c \, (a \cup b)^*}$$

Let L_2 be the language generated by the context-free grammar $G = \{V, \Sigma, P, S\}$, where $\Sigma = \{a, b, c\}$, $V = \{S, A\}$, and the production set P is:

$$S \to AAS \mid \lambda$$
$$A \to a \mid b \mid c$$

(a) Write 5 distinct strings that belong to L_1 and do not belong to L_2 (belong to $L_1 \cap \overline{L_2}$). If such strings do not exist, explain why.

(b) Write 5 distinct strings that belong to L_2 and do not belong to L_1 (belong to $\overline{L_1} \cap L_2$). If such strings do not exist, explain why.

(c) Write 5 distinct strings that belong to L_1 and L_2 (belong to $L_1 \cap L_2$). If such strings do not exist, explain why.

(d) Write 5 distinct strings over alphabet $\{a, b, c\}$ that do not belong to L_1 and do not belong to L_2 (belong to $\overline{L_1} \cap \overline{L_2}$). If such strings do not exist, explain why.

Problem 8 **(a)** Let L_1 be the set of all strings over alphabet $\{a, b, d\}$ that do not contain the substring dba. Write a complete formal definition of a context-free grammar G_1 that generates language L_1. If such a grammar does not exist, explain why.

(b) Let L_2 be the set of all strings over alphabet $\{a, b\}$ that have even length or contain an even number of a's. Write a complete formal definition of a context-free grammar G_2 that generates language L_2. If such a grammar does not exist, explain why.

Problem 9 Let L_1 be the language defined by the regular expression:

$$(\boldsymbol{a} \cup \boldsymbol{ba} \cup \boldsymbol{ca})^* \, (\boldsymbol{b} \cup \boldsymbol{c})$$

Let L_2 be the language generated by the context-free grammar $G = \{V, \Sigma, P, S\}$, where $\Sigma = \{a, b, c\}$, $V = \{S, A, B\}$, and the production set P is:

$$S \to ABABA$$
$$A \to AA \mid a \mid c \mid \lambda$$
$$B \to b$$

(a) Write 5 distinct strings that belong to L_1 and do not belong to L_2 (belong to $L_1 \cap \overline{L_2}$). If such strings do not exist, explain why.

(b) Write 5 distinct strings that belong to L_2 and do not belong to L_1 (belong to $\overline{L_1} \cap L_2$). If such strings do not exist, explain why.

(c) Write 5 distinct strings that belong to L_1 and L_2 (belong to $L_1 \cap L_2$). If such strings do not exist, explain why.

(d) Write 5 distinct strings over alphabet $\{a, b, c\}$ that do not belong to L_1 and do not belong to L_2 (belong to $\overline{L_1} \cap \overline{L_2}$). If such strings do not exist, explain why.

Problem 10 Let L be the language generated by the context-free grammar $G = \{V, \Sigma, P, S\}$, where $\Sigma = \{a, b, c\}$, $V = \{S, A, B, D\}$, and P is:

$$S \to AB \mid D$$
$$A \to AA \mid \lambda \mid abc$$
$$B \to bB \mid \lambda$$
$$D \to aaD \mid bbbD \mid cD \mid \lambda$$

Write a regular expression that defines L. If such regular expression does not exist, prove it.

Problem 11 Write a complete formal definition of a context-free grammar $G = \{V, \Sigma, P, S\}$ over alphabet $\{a, b, c\}$ such that G generates the language of all strings whose length is even or gives remainder 1 if divided by 3. If such grammar does not exist, prove it.

Problem 12 Let L_1 be the language defined by the regular expression:

$$(\boldsymbol{ab})^*$$

Let L_2 be the language generated by the context-free grammar $G = \{V, \Sigma, P, S\}$, where $\Sigma = \{a, b\}$, $V = \{S\}$, and the production set P is:

$$S \to aSa \mid bSb \mid a \mid b \mid \lambda$$

(a) Write a regular expression that defines $L_1 \cap L_2$.
If such regular expression does not exist, explain why.

(b) What is the cardinality of $L_1 \cap L_2$? (If possible, state the exact number. If the set is infinite, specify if it is countable or not.)

(c) Compare the cardinalities of L_1 and L_2, and explain which one (if any) is greater.

(d) Compare the cardinalities of $L_1 \cup L_2$ and $L_1 \cap L_2$, and explain which one (if any) is greater.

Problem 13 **(a)** Let L_1 be a language over alphabet $\{a, b, c, d, e\}$, defined as follows:

$$L_1 = \{a^n b^m c^k d^{m+2} e^{n+3} \mid k, n, m \geq 0\}$$

Write a complete formal definition of a context-free grammar G_1 that generates language L_1. If such grammar does not exist, explain why.

(b) Let L_2 be a language over alphabet $\{a, b, c, d, e\}$, defined as follows:

$$L_2 = \{a^{2n} b^{3n} c^{k+2} d^{3k+1} e^{m+3} a^{2m+5} \mid k, n, m \geq 0\}$$

Write a complete formal definition of a context-free grammar G_2 that generates language L_2. If such grammar does not exist, explain why.

Problem 14 Let:

$$L = \{a^\ell b^{2j} c^k d^{2m} \mid \ell, j, k, m \geq 0\}$$

(a) Write a complete formal definition of a context-free grammar that generates L. If such grammar does not exist, prove it.

(b) Write a regular expression that defines L. If such regular expression does not exist, prove it.

Problem 15 **(a)** Let:

$$L = \{a^i b^j c^k d^m \mid i = j + k \text{ and } m = 2\ell,\ i, j, k, m, \ell \geq 0\}$$

Write a complete formal definition of a context-free grammar that generates L. If such grammar does not exist, prove it.

(b) What is the cardinality of the set of context-free grammars? Answer by giving the exact number (if this set is finite) or by specifying if it is countable or uncountable.

Problem 16 **(a)** Let L be the language defined by the regular expression:

$$(\boldsymbol{a}\,(\boldsymbol{ca} \cup \boldsymbol{da})^*\,\boldsymbol{b}) \cup (\boldsymbol{b}\,(\boldsymbol{ca} \cup \boldsymbol{da})^*\,\boldsymbol{a})$$

Write a complete formal definition of a context-free grammar G_1 that generates language L. If such grammar does not exist, explain why.

(b) Let L be the language defined in part (a).

Write a complete formal definition of a context-free grammar G_2 that generates language L^*. If such grammar does not exist, explain why.

Problem 17 Let L be the set of strings over alphabet $\Sigma = \{a, b, c\}$ defined as follows:

$$L = \{a^m b^k c^\ell \mid m, k, \ell \geq 0 \land m = k + \ell\}$$

1. Write a complete formal definition of a context-free grammar G that generates L. If such grammar does not exist, explain why.
2. List six different strings that belong to $\overline{L}$ (where $\overline{L} = \Sigma \setminus L$). If this is impossible, explain why.

3. List six different strings that belong to

$$\overline{L} \cap \boldsymbol{a^* b^* c^*}$$

(where $\overline{L} = \Sigma \setminus L$). If this is impossible, explain why.

4. State the cardinalities of L and $\overline{L}$, and determine which one is greater (if any.) If possible, give exact numbers, otherwise state if sets are countable or not. Explain your answer briefly.

Problem 18 Let L_1 be the language defined by the regular expression:

$$\boldsymbol{a^* b^*}$$

Let L_2 be the language generated by the context-free grammar $G = (V, \Sigma, P, S)$, where $\Sigma = \{a, b\}$, $V = \{S\}$, and the production set P is:

$$S \to aSb \mid \lambda$$

(a) Write 5 distinct strings that belong to $L_1 \setminus L_2$. If such strings do not exist, explain why.

(b) Write 5 distinct strings that belong to $L_2 \setminus L_1$. If such strings do not exist, explain why.

(c) Write 5 distinct strings that belong to $L_1 \cap L_2$. If such strings do not exist, explain why.

(d) Write 5 distinct strings over alphabet $\{a, b\}$ that belong to $\overline{L_1 \cup L_2}$ (the complement of $L_1 \cup L_2$.) If such strings do not exist, explain why.