Remembering Spherical Trigonometry, To appear in the Math Gazette, March 2016

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1 Spherical Triangles and Spherical Trigonometry

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Although high school textbooks from early in the 20^{th} century show that spherical trigonometry was still widely taught then, today very few mathematicians have any familiarity with the subject. The first thing to understand is that all six parts of a spherical triangle are really angles — see Figure 1.

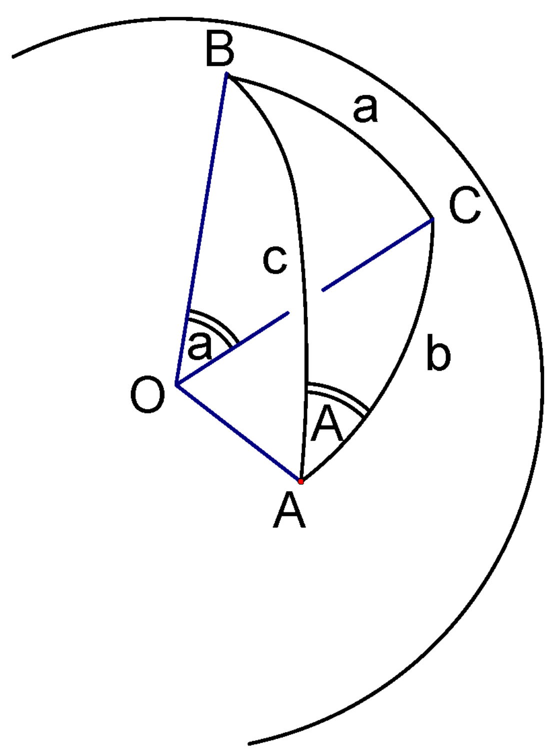


Figure 1: The parts of a spherical triangle.

This shows a spherical triangle ABC on a sphere centered at O. The typical side $a = BC$ is a great circle arc from B to C that lies in the plane OBC; its length is the angle $a = BOC$ subtended at O. Similarly, the typical angle A between the two sides AB and AC is the angle between the planes OAB and OAC.

We have adopted the usual convention that the same letters A, B, C stand for both vertices and angles, and similarly a, b, c for both the edges and their lengths.

Spherical trigonometry, like its planar analogue consists of formulae relating the sides and angles of a triangle. The main difference is that in the spherical versions trigonometric functions are applied to both sides and angles. Fortunately the spherical formulae are so similar to the plane ones that for the most part all we need are simple mnemonics to cover the changes.

There is a long history of such mnemonics from Napier (1614) through Sylvester (1866) to the books of Workman (1912) and Smart (1971). The history of spherical trigonometry is closely bound to the history of astronomy (and through astronomy, to navigation).

The mnemonics in this paper should enable the reader who can already solve problems about plane triangles to do as much for spherical ones. We give no proofs since these would substantially lengthen our paper — most of them can be found in [4].

We believe that we have captured all the relations between sides and angles of a triangle that have ever been taught under just five headings, to which we devote our next five sections: the sine and cosine rules, Napier's Pentagramma Mirificum, the four-part formula, the half angle formulae, and finally the "analogies".

2 The sine and cosine rules

The sine rule of ordinary trigonometry is

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

the common value being $2R$ or $abc/2\Delta$ where R is the circumradius and $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$ the area of the triangle. The spherical sine rule is

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} = \frac{\sin a \sin b \sin c}{2\delta}, \quad \text{where}$$
$$\delta = \sqrt{\sin s \sin(s-a) \sin(s-b) \sin(s-c)}.$$

The number δ does not seem to have a geometrical interpretation — it is related to Δ by *Lexell's* equation* $\frac{\delta}{2} = \cos\frac{a}{2}\cos\frac{b}{2}\cos\frac{c}{2}\sin\frac{\Delta}{2}$, and to the circumradius R by $\delta \tan R = 2\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2}$, but is not a function just of Δ (or of Δ and R). δ is also related to the inradius and exradii by $\delta = \tan r \sin s = \tan r_a \sin(s-a)$, analogous to $\Delta = rs = r_a(s-a)$ in the plane case.

* "Besides Lexell, such a paper could only be written by D'Alembert or me." (Euler)

The cosine rule in spherical trigonometry is

$$\cos a - \cos b \cos c = \sin b \sin c \cos A$$

for a small triangle this reduces to the ordinary rule when we replace $\sin x$ by x and $\cos x$ by $1 - x^2/2$, giving:

$$(1 - a^2/2) \approx (1 - b^2/2)(1 - c^2/2) + bc \cos A$$

To quickly remember the signs, think of an equilateral triangle, for which $\cos a$ is greater than $\cos^2 a = \cos b \cos c$.

Any formula of spherical trigonometry has a polar version, obtained by applying it to the polar triangle, in which sides are replaced by supplements of angles and angles by supplements of sides. So for instance, the polar cosine rule is

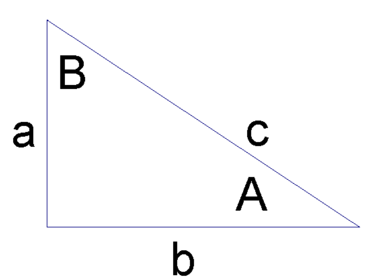
$$-\cos A = \cos B \cos C - \sin B \sin C \cos a$$

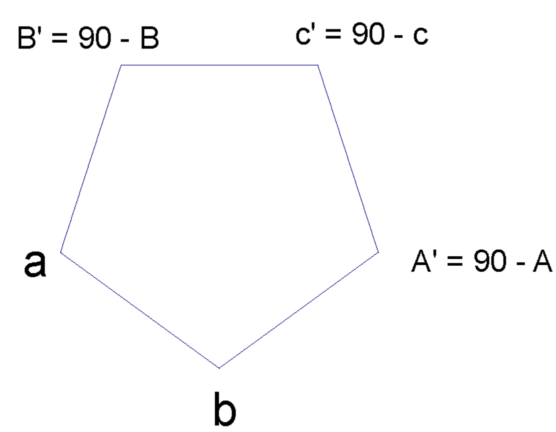
(Mnemonic, in the negative term of the formula for $\cos(B + C)$ insert a factor $\cos a$.)

In this case, as in several others, the corresponding plane formula (obtained by putting $\cos a = 1$) becomes either trivial or *supplemental*, that is, amounts to saying that A is the supplement of $B + C$.

3 Right angled triangles; Napier's Pentagramma Mirificum

The most important special case concerns right-angled triangles, for which there is a marvelous mnemonic due to Napier. Taking C to be $90°$, there are five variable parts, which we imagine in order at the vertices of a pentagon. To quote





Workman's wonderful book [7], "we *mentally write for every part its complement except for those parts which run up to the right angle* (and therefore by way of mnemonic may already be considered to have enough to do with $\frac{1}{2}\pi$), and then apply the formula":

$$\text{s}\textbf{i}\text{n m}\textbf{i}\text{d} \;=\; \text{(prod) c}\textbf{o}\text{s }\textbf{o}\text{pp} \;=\; \text{(prod) t}\textbf{a}\text{n }\textbf{a}\text{dj},$$

which gives a relation for any three parts, since one may be regarded as the mid(dle) one, to which the other two are either opp(osite) or adj(acent).
For example, for the parts a, A', c', the middle is a while A' and c' are the (modified) opposites. So the formula is

$$\sin a = \cos A' \cos c' = \sin A \sin c, \text{ giving } \sin A = \frac{\sin a}{\sin c} \qquad \text{("sin")}$$

For the parts b, A, c, the modified middle is A', adjacent to b and c', so the relation is:

$$\sin A' = \tan b \tan c', \text{ giving } \cos A = \frac{\tan b}{\tan c} \qquad \text{("cos")}$$

while for A, b, a the relation is

$$\sin b = \tan a \; tan A', \text{ or } \tan A = \frac{\tan a}{\sin b}. \qquad \text{("tan")}$$

We recognize "sin", "cos", "tan" as spherical analogues of the usual definitions of sin, cos, tan.

A similar rule works for *quadrantal* triangles, which have a side of 90°, except that the modification is to supplement the two angles adjacent to the quadrant and *cosupplement* the three parts that are not. The cosupplement of an angle θ is $\theta - \pi/2$, the complement of its supplement. (Smart says this rule is the same as Napier's, but that is only true up to sign.)

4 The four-part formula

When we know two angles of a plane triangle, we automatically know the third. The four-part formula of spherical trigonometry substitutes for arguments that rely on this.

For four parts such as A, b, C, a that are consecutive around the triangle, we remember it as:

"cosmic product" = c**ot out**side s**in in**side - c**ot out**angle s**in in**angle
(meaning: outer side, inner side, outer angle, inner angle).

$$\text{for example} \qquad \cos b \cos C = \cot a \sin b - \cot A \sin C$$

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where of course, the "cosmic product" is really the "cos mid product" (the product of cosines of the middle, or inner parts).

5 The half angle and half side formulae

These are:

$$\sin\frac{1}{2}A = +\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b\sin c}} \qquad \cos\frac{1}{2}a = +\sqrt{\frac{\cos(S-B)\cos(S-C)}{\sin B\sin C}}$$

$$\cos\frac{1}{2}A = +\sqrt{\frac{\sin s\sin(s-a)}{\sin b\sin c}} \qquad \sin\frac{1}{2}a = +\sqrt{\frac{-\cos S\cos(S-A)}{\sin B\sin C}}$$

$$\tan\frac{1}{2}A = +\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}} \qquad \tan\frac{1}{2}a = +\sqrt{\frac{-\cos S\cos(S-A)}{\cos(S-B)\cos(S-C)}}$$

Mnemonic for the half angle formulae: The square roots "*cover a multitude of* sins" — if the sins are forgiven (by removing them) we obtain the corresponding planar half angle formulae. Also, the numerator in the **sinus** formula has two **s** m**inuse**s while that in the **cosinus** formula has one **co**mplete s and one **s** m**inus** (these are useful mnemonics for the possibly unfamiliar plane formulae too). The tan formula should be read off from sin / cos.

The half side formulae are the polars of the half angle ones. The sines or cosines involving $\frac{1}{2}$ or s (the *semi*perimeter $\frac{1}{2}(a+b+c)$) are interchanged, but $\sin s$ becomes $-\cos S$, where $S = \frac{1}{2}(A+B+C)$, which is obtuse. Again, the tan one is sin / cos. (Because the half-side formulae determine the sides from the angles, their planar analogues must be trivial or supplemental. In this case, since $S = \pi/2$, $\cos(S-B) = \sin B$ and $\cos S = 0$.)

6 The so-called "analogies"

The formulae that are most in need of mnemonics are the "analogies" of Napier and Delambre. The word "analogy" itself deserves some explanation. Perhaps its best modern translation is "proportionality," written $P : Q :: R : S$ meaning that P is to Q as R is to S. Nowadays we express this as an equality of ratios $P/Q = R/S$. After all, the proportionality $P : Q :: R : S$ expresses an analogy in the surviving sense of that word between the relation of P to Q and that of R to S.

The typical "analogy" in the historical sense equates two ratios of trigonometric functions, one of the sides of a triangle and the other of its angles. There are two sets of four analogies, an older set due to Napier, called Napier's analogies ever since his day, and a newer set due to Delambre. However, the history of Delambre's analogies is rather tangled, and they are still often attributed to Gauss (eg. by Workman). Delambre's analogies have been known as Delambre's formulae, Gauss' formulae, and even the Gauss-Mollweide formulae. Sylvester proposed in 1866 that they be called "analogies", by analogy with Napier's.

In his 1873 paper, Isaac Todhunter takes great pains to determine the correct attribution and comes down firmly on the side of Delambre.

Delambre himself said in his *Astronomie* that he had published them in the *Connaissance des Tems* for 1808, but it was really in the volume for 1809, which was actually available in April 1807! (Like later Almanacs and Ephemerides the *Connaissance des Tems* made prognostications and astronomical predictions for a coming year and so was published in advance.) Mollweide published the analogies in 1808 and Gauss in his *Theoria Motus Corporum Coelestium* of 1809. The interest in these years was prompted by the immediately previous discovery of the first four asteroids: Ceres on 1^{st} January 1801, Pallas on 28^{th} March 1802, Juno on 1^{st} September 1804, and Vesta on 29^{th} March 1807. (The fifth asteroid, Astraea, had to wait until 8^{th} December 1845.)

Sylvester also remarks on the difficulty of remembering the analogies and gives his own mnemonic. Our interest in this subject stems from an improved mnemonic that we found written by an anonymous previous owner (who we'll call 'the student') in a secondhand copy of Smart's *Textbook on spherical astronomy* (5th edition, 1971).

Smart deals with Delambre's analogies on page 22, and at the foot of this page, the student has written the following mnemonic for them:

$$\frac{\text{cos-or-sinus } \textit{semi}(\textit{side} \text{ Plus-or-Minus } \textit{side})}{\text{Cos-or-Sinus } \textit{semi}(\textit{ANGle} \text{ plus-or-minus } \textit{ANGle})} = \frac{\text{same}(\textit{semi side})}{\text{chANGed}(\textit{semi ANGle})}$$

where the capitalization indicates that the ambiguous sign in the numerator corresponds to the rhyming ambiguous function in the denominator, and vice versa.

On page 23 we find the student trying to find mnemonics for Napier's analogies (which Smart discusses there). The student has filled all the empty spaces on that page with trial mnemonics of other kinds (one of which involves "cot(*côte*)").

In the student's mnemonic 'same' and 'chANGed' on the right hand side refer to the functions on the left. In our revision, these words in a denominator refer to the function in the numerator, following the arrangement in Sylvester's (otherwise worse) mnemonic.

This simple alteration makes the student's mnemonic give nine "analogies". In the typical one, called "trig = TRIG", trig represents a trigonometric function sin, cos, tan to be applied to sides, and TRIG another one, SIN, COS, TAN, to be applied to angles. (This matches the convention that angles are written in capital letters.) Each trigonometric function (trig or TRIG) has an image 'signum' (sig or SIG) which is $-$ for sin, $+$ for cos, and $-/+$ for tan. Our new mnemonic has shape "trig(SIG) = TRIG(sig)", or in full:

$$\frac{\text{trig } \textit{semi}(\mathbf{s}\textit{ide} \text{ SIG } \mathbf{s}\textit{ide})}{\mathbf{s}\text{ame trig } \textit{semi}(3^{rd}\mathbf{s}\textit{ide})} = \frac{\text{TRIG } \textit{semi}(\textit{ANGLE} \text{ sig } \textit{ANGLE})}{\text{ch}\mathbf{ANG}\text{ed TRIG } \textit{semi}(3^{rd}\textit{ANGle})}.$$

Note that the "signum" on one side of the equation corresponds to the "trig'nom" (trigonometric function) on the other. The "chANGge" of a TRIG is its cofunction. Also trig $\frac{1}{2}(\alpha -/+ \beta)$ is to be interpreted as trig $\frac{1}{2}(\alpha - \beta)/\text{trig } \frac{1}{2}(\alpha + \beta)$, already a ratio. In this case the denominator in the mnemonic should be omitted since it takes the form $f(\frac{1}{2}\gamma)/f(\frac{1}{2}\gamma) = 1$, for some function f.

If neither tan nor TAN is mentioned, we have the student's mnemonic for the Delambre analogies; if just one appears, our new mnemonic for the Napiers (if TAN), or their polars (if tan); while finally the identity tan = TAN gives the formula

$$\frac{\tan\frac{1}{2}(a-b)}{\tan\frac{1}{2}(a+b)} = \frac{\mathrm{TAN}\frac{1}{2}(A-B)}{\mathrm{TAN}\frac{1}{2}(A+B)}$$

which we call "the tangent analogy". This appears, though not by this name, in several 19^{th} century works (it is equivalent to the law of sines in the form $\frac{\sin a}{\sin b} = \frac{\sin A}{\sin B}$, which is already an analogy in the historical sense).

Any difficulties in following our mnemonic should be cleared up by the following table, which lists all nine cases.

Delambre's Analogies		Napier's Analogies
sin = SIN $\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}c} = \frac{\mathrm{SIN}\frac{1}{2}(A-B)}{\mathrm{COS}\frac{1}{2}C}$	sin = COS $\frac{\sin\frac{1}{2}(a+b)}{\sin\frac{1}{2}c} = \frac{\mathrm{COS}\frac{1}{2}(A-B)}{\mathrm{SIN}\frac{1}{2}C}$	sin = TAN $\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)} = \frac{\mathrm{TAN}\frac{1}{2}(A-B)}{\mathrm{COT}\frac{1}{2}C}$
cos = SIN $\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}c} = \frac{\mathrm{SIN}\frac{1}{2}(A+B)}{\mathrm{COS}\frac{1}{2}C}$	cos = COS $\frac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}c} = \frac{\mathrm{COS}\frac{1}{2}(A+B)}{\mathrm{SIN}\frac{1}{2}C}$	cos = TAN $\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)} = \frac{\mathrm{TAN}\frac{1}{2}(A+B)}{\mathrm{COT}\frac{1}{2}C}$
tan = SIN $\frac{\tan\frac{1}{2}(a-b)}{\tan\frac{1}{2}c} = \frac{\mathrm{SIN}\frac{1}{2}(A-B)}{\mathrm{SIN}\frac{1}{2}(A+B)}$	tan = COS $\frac{\tan\frac{1}{2}(a+b)}{\tan\frac{1}{2}c} = \frac{\mathrm{COS}\frac{1}{2}(A-B)}{\mathrm{COS}\frac{1}{2}(A+B)}$	tan = TAN $\frac{\tan\frac{1}{2}(a-b)}{\tan\frac{1}{2}(a+b)} = \frac{\mathrm{TAN}\frac{1}{2}(A-B)}{\mathrm{TAN}\frac{1}{2}(A+B)}$
Polars of Napier's Analogies		Tangent Analogy

If trig is sin or tan, by omitting this function we obtain a useful planar analogue, but if trig is cos the planar form is merely supplemental.

7 Hyperbolic Trigonometry

Many professional mathematicians are currently interested in hyperbolic geometry, and so in hyperbolic trigonometry. Fortunately, the formulae for hyperbolic trigonometry are easily found from the spherical ones as follows: Replace the trigonometric functions of the sides by hyperbolic functions and apply Osborn's rule that deduces hyperbolic identities from the better known trigonometric ones. As we remarked in our paper [1], Osborn's statement [3]: "change the sign of any term that contains a product of sinhs" was insufficiently precise.

[He should have told us to change the sign for each successive pair of sinhs, and so to multiply by $(-1)^k$ any term that contains $2k$ or $2k+1$ sinhs.] For example, applied to the (spherical) cosine rule this gives the hyperbolic cosine rule:

$$\cosh a = \cosh b \cosh c - \sinh b \sinh c \cos A.$$

In Osborn's rule it is to be understood that the formulae should be thought of as a polynomial identity in sines and cosines. So for instance the four-part formula:

$$\cos b \cos C = \cot a \sin b - \cot A \sin C$$

.

becomes

$$\cosh b \cos C = \coth a \sinh b - \cot A \sin C$$

.

Note that the first term on the right has one sine in the numerator and one in the denominator and so keeps the same sign in the hyperbolic form, while the second involves only angles, so is unchanged.

Our mnemonics should enable you to expand your knowledge of plane trigonometry to find whatever you need of spherical and hyperbolic trigonometry. However, we must confess that in what might be considered a negative review of this paper, Augustus de Morgan wrote (about Napier's pentagram) that "mnemonical formulas ... only create confusion instead of assisting the memory" [2]. But times have changed.

References

[1] J. H. Conway and A. Ryba, Fibonometry *The Mathematical Gazette,* **97** issue 540, (2013), 494-495.

[2] A. de Morgan, On the invention of circular parts *Philosophical Magazine,* Series 3 Volume 22,(1843) 350-353.

[3] G. Osborn, Mnemonic for hyperbolic formulae, *The Mathematical Gazette,* **2** issue 34, (1902) p. 189.

[4] W. M. Smart, Text-book on Spherical Astronomy, Cambridge University Press, 5^{th} edition, (1971) (the first edition was 1928).

[5] J. J. Sylvester, Note on a memoria technica for Delambre's, commonly called Gauss's theorems *Philosophical Magazine,* Series 4, Volume 32, (1866) 436–438.

[6] I. Todhunter, Note on the history of certain formulae in spherical trigonometry *Philosophical Magazine,* Series 4, Volume 45, (1873) 98—100.

[7] W. P. Workman, Memoranda Mathematica, Clarendon Press, Oxford, 1912