QUEENS COLLEGE **Department of Mathematics**
Math 618 **Second Midterm Exam Spring 2018 05.16.18**

Numerical Answers

6.15pm – 7.30pmWednesday, May 16, 2018

Problem 1. The points A, B and C in the upper half plane model of the the hyperbolic plane have coordinates $A = (0, 3)$, $B = (8, 3)$, $C = (-4, 5)$.

(a) Find the hyperbolic distance between A and B. Your answer should involve only a logarithm.

$\boxed{\log 9}$

(b) Find the cosine of the hyperbolic angle BAC. Your answer should be fully simplified.

$\boxed{7/25}$

Problem 2. A hyperbolic triangle ABC has angles $A = \pi/6$ and $B = \pi/6$. Its area is $\pi/6$. The sides opposite angles A, B and C have hyperbolic measures a, b and c. The answers to this question might involve square roots, but no other functions.

(a) Find the angle C of the triangle.

$\boxed{\pi/2}$

(b) Find $\cosh a$.

$\boxed{\sqrt{3}}$

(c) Find $\sinh b$.

$\boxed{\sqrt{2}}$

(d) Find $\tanh c$.

$\boxed{\sqrt{8}/3}$

Problem 3. Let $A = (0, 6), B = (6, 6), C = (6, 12)$ and $D = (0, 12)$ be points in the upper half plane model of the hyperbolic plane. The answers to this question should be written in terms of inverse tangents but no other functions.

(a) Find the hyperbolic area of the rectangular shape formed by the Euclidean segments AB, BC, CD and DA.

$\boxed{1/2}$

(b) Find the tangents of the angles of the hyperbolic quadrilateral formed by joining A, B, C and D in this order by hyperbolic geodesics.

$\boxed{2, 2, -4, -4}$

(c) Find the hyperbolic area of the hyperbolic quadrilateral of (b). Write you answer as a combination of values of inverse tangents. (Hint: Break the quadrilateral into two triangles by a diagonal. Now how do the angles of (b) allow you to find the sum of the areas of these triangles?)

$\boxed{2\tan^{-1} 4 - 2\tan^{-1} 2}$

Problem 4. (a) Suppose that l is a hyperbolic line and P is a hyperbolic point. Explain why there must be a hyperbolic line through P that meets l at an angle of $90°$.

Answer: Let m be the hyperbolic line that joins P to its image under hyperbolic reflection in l. Then m is a perpendiular to l that passes through P.

(b) Explain why there can only be one hyperbolic line through P that meets l at an angle of $90°$.

Answer: If there were 2 perpendiculars m and n then l, m and n would be three sides of a hyperbolic triangle with 2 right angles. This triangle would have an angle sum larger than $180°$, a contradiction.

The distance from a hyperbolic point P to a hyperbolic line l is the *perpendicular distance*. To find this, we let m be the hyperbolic line through P that is perpendicular to l, let Q be the intersection of l and m and calculate the hyperbolic distance between P and Q.

(c) Let y be the hyperbolic line represented in the half plane model by the y-axis. Let P be any point of the half plane. Let δ be a Euclidean dilation centered at the origin. Explain why the points P and $\delta(P)$ are at the same distance from y.

Answer: Let m be the Euclidean circle that represents the hyperbolic line through P that is perpendicular to y. Then m is centered at the origin. Let Q, R and S be the intersections of m with the negative x-axis, y, and the positive x-axis. The (hyperbolic) distance from P to y is $|\log((PQ \times RS)/(PS \times RQ))|$. This is equal to the log of the dilated cross ratio, but by the same reasoning this represents the distance from $\delta(P)$ to y.

(d) Suppose that g is a circle that represents a geodesic in the half plane model of hyperbolic geometry. Describe the locus of points that are at distance 1 from g.

Answer: According to (c) the locus of points at distance 1 from y is a line through the origin. We can apply a sequence of hyperbolic reflections that transform y to g. The locus of points at (hyperbolic) distance 1 from y is transformed to the required locus. Since the transformations that we apply are inversions (and refletions), the locus is a (the arc) of circle that meets the x-axis in the 2 points that lie on g.

(Extra Credit) How (if at all) would the answer to (d) change in the disk model of hyperbolic geometry?

The locus would either be an arc of a circle or a straight line (that meets the unit circle at its two points of intersection with g).