Practice problems on inversive geometry.

Problem 1 Let l be a line that does not pass through the origin. Consider the transformation of inversion in the unit circle (centered at the origin). Let l^* be the inverse of l. Prove that l^* is a circle.

Problem 2 Suppose that P and Q are points. Prove that the locus of all points R with $QR/PR = 2$ is a circle.

Problem 3 Let C be a circle that does not pass through the origin. Consider the transformation of inversion in the unit circle (centered at the origin). Let C^* be the inverse of C. Prove that C^* is a circle.

Problem 4 Find a formula for $F(z)$ where F is a Mobius transformation with $F(-2i) = 0$, $F(2) = 2$, and $F(1-i) = 1-i$. (Write your answer as a function $F(z)$ that has the form $\frac{az+b}{cz+d}$.)

Problem 5 Find a formula for $F(z)$ where F is a Mobius transformation with $F(0) = -2i$, $F(2) = 2$, and $F(1-i) = 1-i$. (Write your answer as a function $F(z)$ that has the form $\frac{az+b}{cz+d}$.)

Problem 6 (a) Do the four points 2, -2, $2+3i$, and $1+4i$ lie on a circle. (You must explain your answer to get any credit!)

(b) Describe the image curve that is obtained when the y-axis (the imaginary axis) is transformed by using the Mobius transformation:

$$M(z) = \frac{iz+1}{z+1}$$

(You must completely describe the image curve. It is not enough just to say that it is a line or to say that it is a circle. You should give details of the slope, center or other features that completely describe the curve.)

Problem 7 (a) Do the four points 2, -2, $2+3i$, and $1+3i$ lie on a circle. (You must explain your answer to get any credit!)

(b) Describe the image curve that is obtained when the y-axis (the imaginary axis) is transformed by using the Mobius transformation:

$$M(z) = \frac{-iz+1}{z+1}$$

(You must completely describe the image curve. It is not enough just to say that it is a line or to say that it is a circle. You should give details of the slope, center or other features that completely describe the curve.)

Problem 8 Find a formula for $F(z)$ where F is a Mobius transformation with $F(1) = \infty$, $F(2) = 1$, and $F(i) = 2$. (Write your answer as a function $F(z)$ that has the form $\frac{az+b}{cz+d}$.)

Problem 9 Find a formula for $F(z)$ where F is a Mobius transformation with $F(1) = \infty$, $F(2) = 0$, and $F(i) = 2$. (Write your answer as a function $F(z)$ that has the form $\frac{az+b}{cz+d}$.)

Problem 10 Find a formula for $F(z)$ where F is a Mobius transformation with $F(1) = \infty$, $F(2) = 1$, and $F(i) = 3$. (Write your answer as a function $F(z)$ that has the form $\frac{az+b}{cz+d}$.)

Problem 11 Suppose that the points P and Q are inverses with respect to a circle Σ.

(i) Let f be an inversive transformation that transforms Σ to a circle. Prove that f transforms P and Q to points that are inverse with respect to $f(\Sigma)$.

(ii) Suppose that g is an inversive transformation that transforms Σ to a line. Explain the relationship between $g(P)$ and $g(Q)$.

Problem 12 Suppose that a pair of orthogonal circles meet at the points O and P. Suppose that a third circle C intersects the first circle orthogonally at X and the second circle orthogonally at Y.

(i) Consider an inversion centered at O. Draw a diagram to indicate the images of the three circles and mark the inverses P', X', and Y' of the three points P, X, and Y.

(ii) Explain briefly why $X'Y'$ must be a diameter of the circle that passes through P', X', and Y'.

(iii) Determine the angle between the circle that passes through P, X, and Y and the circle C. (Hint: Can you find the angle between the inverses of these curves.)

Problem 13 Suppose that P is a point outside the circle Σ. Let O be the center of Σ. Let the tangents from P meet Σ at X and Y. Let XY meet OP at Z.

Prove that P and Z are inverse points with respect to Σ.

Problem 14 Find a formula for $F(z)$ where F is a Mobius transformation with $F(\infty) = 1$, $F(1) = 2$, and $F(4) = i$. (Write your answer as a function $F(z)$ that has the form $\frac{az+b}{cz+d}$.)

Problem 15 Suppose that three circles, a, b, and c all pass through the points P and Q. Let d be a circle orthogonal to both a and b.

(i) Consider an inversion centered at P. Draw a diagram to indicate the images of the four circles a, b, c, and d. Mark the inverse Q'.

(ii) Prove that d must be orthogonal to c.

(iii) If the circles a, b and c are considered as fixed, how many different circles could play the role of d?

Problem 16 Let P, Q, and R be the points $-1+i$, $1-i$, $-1-2i$ of the extended complex plane $\widehat{C}$.

(i) Find a formula for a Mobius transformation $F(z)$ with $F(0) = P$, $F(1) = Q$, and $F(\infty) = R$.

(ii) Give a short calculation which proves that the point -2 lies on the generalized circle through P, Q, and R.

Problem 17 Let C_1, C_2, and C_3 be three circles that touch each other externally in pairs. (In other words, C_1 touches C_2 at a point Z, C_2 touches C_3 at a point X, and C_3 touches C_1 at a point Y.)

Let D be a small circle in the region between C_1, C_2, and C_3 that touches all three circles.

(i) Draw a diagram of the four circles C_1, C_2, C_3 and D.

(ii) Draw a diagram that shows the images of the four circles under an inversion centered at the point X where C_2 and C_3 touch. Explain briefly why the images are as you have shown them.

(iii) Prove that there is a generalized circle G that touches both C_2 and C_3 at X and is orthogonal to both C_1 and D.

(Extra credit:) How many generalized circles meet the requirements on G? (Why?) In what circumstances is G a straight line?

Problem 18 Find a formula for $F(z)$ where F is a Mobius transormation with $F(1) = i$, $F(2) = -1$, and $F(3) = -i$.

Problem 19 Define the cross ratio $(a, b : c, d)$ of four points a, b, c, d. Prove that the cross ratio is a Mobius invariant.

Problem 20 Assume that the Mobius transformation $F(z) = \frac{-iz}{z-2i}$ transforms the circle C with equation $x^2 + (y-1)^2 = 1$ to the x-axis.

(i) Write down a formula for the inverse Mobius transformation F^{-1}.

(ii) Let R_x be the transformation of reflection across the x-axis. How can inversion across C be obtained from R_x, F, and F^{-1}.

(iii) Find a formula for the transformation of inversion across C.

(iv) Find the inverse of i across the circle C.

Problem 21 Let C be the circumcircle of the triangle with vertices at 1, i and $-i$. Find a formula for $F(z)$ where F is a Mobius transformation that transforms C to the x-axis.

Problem 22 Given two circles C_1 and C_2 and a point P. Prove that there is a cline C that passes through P and is orthogonal to both C_1 and C_2.

Problem 23 Find a formula for $F(z)$ where F is a Mobius transformation with $F(2) = 5$, $F(4) = 0$, and $F(8) = 3$.