QUEENS COLLEGE **Department of Mathematics**
Math 618 **First Midterm Exam** **Spring 2018** **03.28.18**

Solutions

Wednesday, March 28, 2018

Problem 1. A spherical triangle has an area of 45°. Two of its angles are 60° and 45°.

(a) Find the third angle of the triangle.

Answer:

The area is 180° less than the angle sum, which must be 225°. Hence the third angle is 120°.

(b) Find the cosine of the (arc) length of the side opposite the angle angle you just found.

Answer:

Let $A = 120°$, $B = 60°$ and $C = 45°$ be the angles of the triangle. The *polar law of cosines* gives:

$$\cos A + \cos B \cos C = \sin B \sin C \cos a$$

. Hence:

$$-\frac{1}{2} + \frac{1}{2} \times \frac{1}{\sqrt{2}} = \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} \cos a$$

.

This gives

$$\cos a = \frac{1 - \sqrt{2}}{\sqrt{3}}$$

.

Problem 2. A plane is to fly a geodesic route from Sydney, Australia with latitude $30°S$ and longitude $150°E$ to Bogota, Colombia with latitude $0°N$ and longitude $75°W$. The radius of the earth is $R \approx 4000$ miles.

(a) Draw a diagram of a spherical triangle with vertices B, N, S that represent Bogota, the north pole and Sydney. Mark values for those sides and angles that follow immediately from the latitudes and longitudes.

Answer:

The following diagrams show the triangle NBS and also its polar triangle YZX. The polar triangle has a right angle and therefore has a Napier pentagram which is also shown. Note that travelling towards the east from S to B gives an angle $N = 135°$ whereas travelling towards the west would give an angle $360° - 135° = 225°$. The easterly direction must be used for the geodesic.

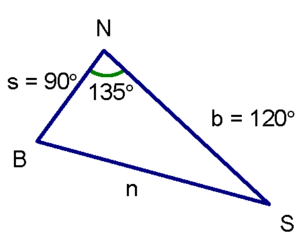


Diagram of the triangle NBS

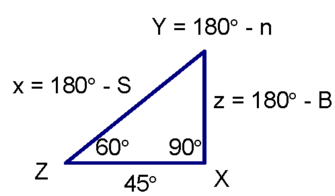
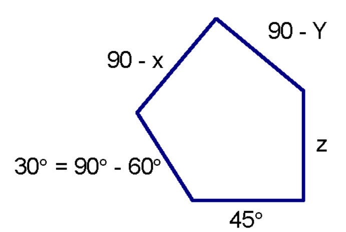


Diagram of the polar triangle YZX of NBS



Napier's pentagram for the polar triangle

(b) Give a formula for the length of the flight. Your formula should be written as a multiple of R and this multiple should be left in terms of π and a single inverse trig functions. (Do not use a calculator to give an exact answer.)

Answer:

We need to find n in triangle NBS. This is the supplement of Y in the polar triangle. Napier's pentagram gives

$$\cos Y = \sin(90 - Y) = \cos 30 \ \cos 45 = \sqrt{\frac{3}{8}}$$

.

We deduce that $n = \pi - \cos^{-1}\sqrt{3/8}$ and therefore the distance from Syndney to Bogota is $R(\pi - cos^{-1}\sqrt{3/8})$.

(c) Give the direction in which the plane should leave from Sydney. Write this as a number of degrees east or west of north. The number of degrees should be written as a single inverse trig function and you should specify whether the flight is directed east or west of north. (Do not use a calculator to give an exact answer.)

Answer:

We need to find angle S or its supplement x. Napier's pentagram gives $\sin 30 = \tan 45\ \cot x$. Hence $\tan x = \tan 45/\sin 30 = 2$.

We deduce that the angle the plane should leave from Sydney is $180° - \tan^{-1} 2$ to the East of North.

Problem 3. A (badly drawn) figure of eight is formed by a circle with center $(0, -2)$ and radius 3 and a circle with center $(0, 3)$ and radius 2. An inversion transforms the figure of eight into two parallel lines that are 5 units apart.

(a) What is the center of the inversion? (Write the coordinates of a point for your answer.)

Answer: If the circles are both transformed to lines the center of inversion must be a point that lies on both. The two circles touch at $(0, 1)$ and this must be the center of inversion.

(b) What is the radius of inversion?

Answer:

The figure of eight curve is inverted to two lines parallel to the x-axis (because the common diameter of its circles is the y-axis).

If r is the radius of inversion then the points $(0, 5)$ and $(0, -5)$ which lie on the figure of eight curve invert to points on the y-axis that are $r^2/4$ and $r^2/6$ units above and below the center of inversion. Therefore, $r^2/4 + r^2/6 = 5$. Hence, $r^2 = 12$.

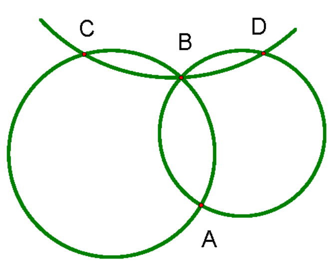
We deduce that the radius of inversion is $\sqrt{12}$.

Problem 4. Suppose that a diagram is made from two orthogonal circles that meet at points A and B and another circle that passes through B and meets the original pair of circles at two other points C and D.

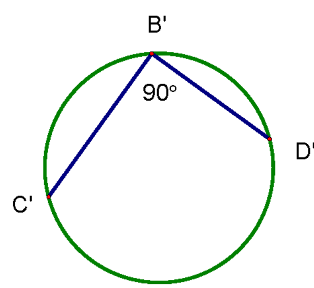
(a) Draw the figure obtained by applying an inversion centered at A to the diagram.

(b) Draw the figure obtained by applying an inversion centered at B to the original diagram.

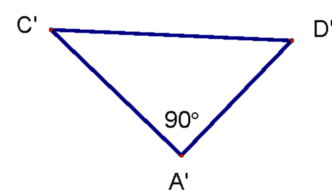
Answer: The following diagrams show the original configuration and its inversions relative to centers at A and B.



Original diagram



Inversion with center A



Inversion with center B

(c) Write a definition for the cross ratio of 4 points $(U, V : W, X)$.

Answer:

$$(U, V : W, X) = \frac{UW \times VX}{UX \times VW}$$

.

(d) Suppose that A, B, C and D are the points mentioned in the original diagram. Prove that:

$$(A, C : D, B)^2 + (A, D : C, B)^2 = 1$$

.

Answer:

Consider an inversion with center B. Recall that the distance bewteen a pair of inverse points is given by:

$$X'Y' = \frac{r^2 XY}{BX \times BY}$$

.

However, as above the inverse diagram is a right triangle. Pythagoras gives:

$$(A'C')^2 + (A'D')^2 = (C'D')^2$$

.

Hence,

$$\frac{AC^2}{BA^2 BC^2} + \frac{AD^2}{BA^2 BD^2} = \frac{CD^2}{BC^2 BD^2}$$

.

If we divide through by the right hand side $\frac{CD^2}{BC^2 BD^2}$, we get: $(A, C : D, B)^2 + (A, D : C, B)^2 = 1$.