QUEENS COLLEGE — Department of Mathematics

Math 618 — Second Midterm Exam — Spring 2013 — 05.13.13

Solutions

4.30pm – 5.45pm, Monday, May 13, 2013

Problem 1. **CHOOSE ONE OPTION ONLY, there is no extra credit for doing both. Either:**

State and prove a theorem giving the area of a triangle in hyperbolic geometry. You should include a careful statement of any results that you assume. A complete proof must include the correct evaluation of at least one integral!

Answer: See Theorem 7.21 and Lemma 7.2.2 on page 93 of the text.

Or:

A hyperbolic triangle has vertices: $A = (0, \sqrt{34})$, $B = (0, 4)$ and $C = (3, 5)$. Find the cotangents of the angles and the lengths of the sides of the triangle. (Write your answers into this grid and give your work below.)

cot(A) = 0	cot(B) = $\frac{3}{4}$	cot(C) = $\frac{5}{3}$
side a = ln(2)	side b = $\ln(\frac{5}{\sqrt{34}-3})$	side c = $\ln(\frac{\sqrt{34}}{4})$

Answer:

The angles A, B, C are $90°$, β, γ in the Figure.

We have $c = ln(\frac{\sqrt{34}}{4})$ (from the formula for the length of a straight geodesic). The formula for the length of a curved geodesic gives $a = ln(\frac{csc(180-\beta)-cot(180-\beta)}{csc90-cot90}) = ln(\frac{csc\beta+cot\beta}{1-0}) = ln(\frac{1+cos\beta}{sin\beta}) = ln2$, and $b = ln(\frac{csc90-cot90}{csc(90-\gamma)-cot(90-\gamma)}) = ln(\frac{1}{sec\gamma-tan\gamma}) = ln(\frac{5}{\sqrt{34}-3})$.

Problem 2. Suppose that A, B, C, and D are the vertices $(0, 0)$, $(2, 0)$, $(2, 2)$, and $(0, 2)$ of a square.

(a) The transformation $\tau_{BC} \circ R_{A,90}$ is a rotation $R_{X,\alpha}$. Find the coordinates of X and the angle α.

Answer:

$X = (-1, 1)$ and $\alpha = 90°$.

(b) The transformation $\gamma_{AD} \circ \gamma_{AB}$ is a rotation. Identify its center and angle of rotation.

Answer:

The center is $(-1, 1)$ and the angle is $180°$.

Problem 3. A hyperbolic isosceles triangle ABC has sides with lengths a, b, c that satisfy $\cosh(a) = 4$, $\cosh(b) = \sqrt{3}$, $\cosh(c) = \sqrt{3}$.

(a) Find $\sinh(b)$. (You must simplify this answer to get full credit!)

Answer:

$\sinh^2 b = \cosh^2 b - 1 = 3 - 2 = 2$. Thus $\sinh b = \sqrt{2}$.

(b) Find $\cos(\alpha)$, where α is the angle of the triangle opposite the side a.

Answer:

The law of cosines states that:

$$\cosh a = \cosh b \cosh c - \sinh b \sinh c \cos\alpha$$

Substituting for the hyperbolic functions we obtain:

$$4 = \sqrt{3}\sqrt{3} - \sqrt{2}\sqrt{2}\cos\alpha$$

Hence $\cos\alpha = \frac{-1}{2}$. (It follows that $\alpha = 120°$ and $\sin\alpha = \sqrt{3}/2$.)

(c) Let CH be the perpendicular geodesic from the vertex C to the opposite side c. Let h be the hyperbolic length of CH. Find $\sinh(h)$.

Answer:

From the right triangle CHA we have

$$\frac{\sqrt{3}}{2} = \sin(180 - \alpha) = \frac{\sinh h}{\sinh b}$$

We deduce that

$$\sinh h = \frac{\sqrt{3}\sinh b}{2} = \frac{\sqrt{3}}{\sqrt{2}}$$

Problem 4. Calculate the hyperbolic area of the region bounded by the y-axis, the line $y = x$, the circle centered at the origin with radius 1, and the circle centered at the origin with radius 2.

Answer:

If R is the given region and $h(R)$ is its hyperbolic area then:

$$h(R) = \int\int_R \frac{dx\ dy}{y^2} = \int\int_R \frac{rdr\ d\theta}{r^2 sin^2\theta} = (\int_{r=1}^{2} \frac{dr}{r})(\int_{\theta=\pi/4}^{\pi/2} csc^2\theta d\theta) = (ln2 - ln1)(-cot(\pi/2) + cot(\pi/4)) = ln2$$

Problem 5. Suppose that two circles p and q intersect at A and B. Let l and m be the diameter lines of p and q through B. Suppose that l meets q at C and that m meets p at D. Prove that the line AB is a diameter line of the circle BCD.

(For partial credit, draw a diagram, choose a center of inversion and draw a careful diagram of the inverted configuration.)

Answer:

Suppose that in inversion centered at B transforms A, C, p, q, l, m to A', C', p', q', l', m'. Then p', q', l', m' are lines and $p' \cap q' = A'$, $l' \cap m' = B$, $l' \cap q' = C'$, $m' \cap p' = D'$. Moreover, $l' \perp p'$ and $m' \perp q'$. It follows that p' and q' are altitudes of triangle $D'C'B$. Therefore A' is the orthocenter of this triangle. Hence BA' is the third altitude of the triangle and is perpendicular to the side $C'D'$. Inverting back, we deduce that AB is orthogonal to the circle BCD. In other words AB is a diameter line of the circle.