Computer Generation of Incidence Theorems

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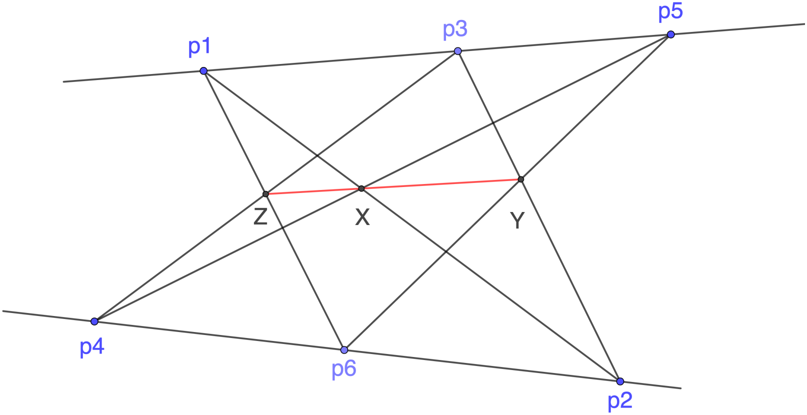
Pappus and Desargues

The two standard incidence theorems are due to Pappus and Desargues. They exhibit 9_3 and 10_3 configurations and apply to certain arrangements of 6 initial points.

Pappus

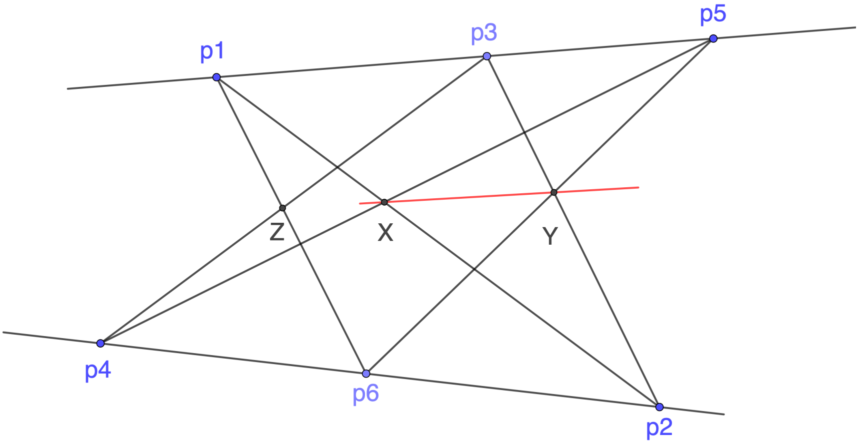
Theorem (**Pappus**)

If the vertices of a hexagon lie alternately on 2 lines then the 3 points of intersection of opposite sides of the hexagon are collinear.



Is Pappus a theorem?

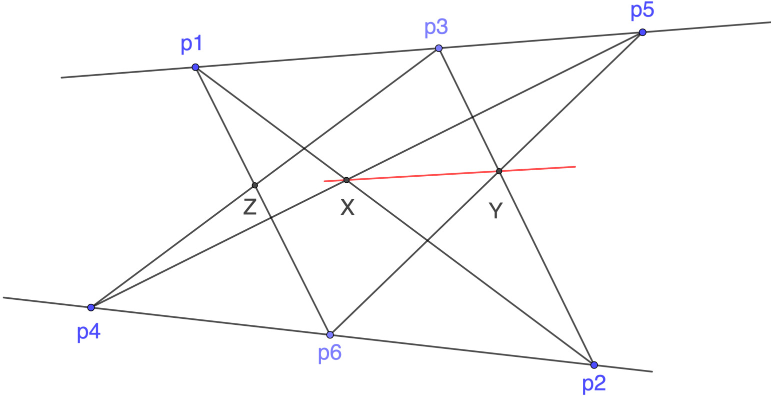
Pappus said so 1700 years ago.



We agree. We don't expect X, Y and Z to line up.

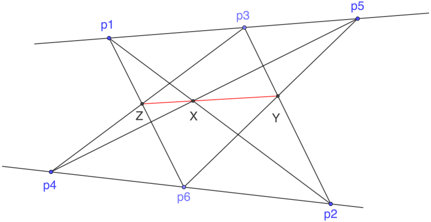
We can quantify our belief that Pappus is a theorem

The initial points $1, 2, 3, 4, 5, 6$ are ingredients.
Other objects have recipes: X is $\langle [1\ 2]\ [4\ 5] \rangle$
The *Pappus line* XY is $[\langle [1\ 2]\ [4\ 5] \rangle\ \langle [3\ 2]\ [6\ 5] \rangle]$



Ingredient permutations that preserve a recipe

Pappus line XY is $[\langle[1\ 2]\ [4\ 5]\rangle\ \langle[3\ 2]\ [6\ 5]\rangle]$
14.25.36 maps it to $[\langle[4\ 5]\ [1\ 2]\rangle\ \langle[6\ 5]\ [3\ 2]\rangle]$
13.46 maps it to $[\langle[3\ 2]\ [6\ 5]\rangle\ \langle[1\ 2]\ [4\ 5]\rangle]$
Combinatorial stabilizer of the recipe S_C has size 4.



Geometric stabilizer also includes 135.246 because $XY = YZ = ZX$.
Geometric stabilizer S_G has size 12.
Any recipe where $S_C \neq S_G$ gives an incidence theorem.

Algorithm

Choose a set of input objects.
Generate recipes recursively.
Compute the stabilizers S_C and S_G for the recipes.
Whenever the stabilizers differ output a theorem.

Are there theorems with 5 input points?

Generation 1 (Point types: 1, Line types: 0)

Generation 2 (Point types: 1, Line types: 1)

Generation 3 (Point types: 2, Line types: 1)

Generation 4 (Point types: 2, Line types: 4)

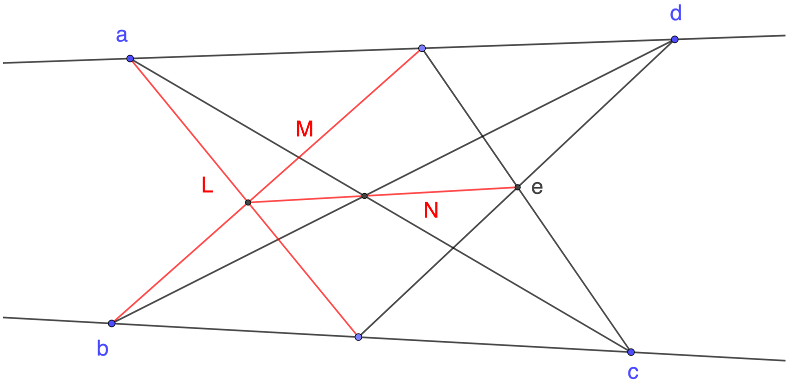
Generation 5 (Point types: 25, Line types: 4)

20: $\langle\ [a\ \langle [bc][de] \rangle]\ [b\ \langle [ad][ce] \rangle]\ \rangle : |S_C| = 2,\ |S_G| = 6,\ abe$

10: $\langle\ [\langle [bd][ec] \rangle\ \langle [be][dc] \rangle]\ [\langle [ad][ec] \rangle\ \langle [ae][dc] \rangle]\ \rangle : |S_C| = 4 \ldots$

The computer's first theorem.

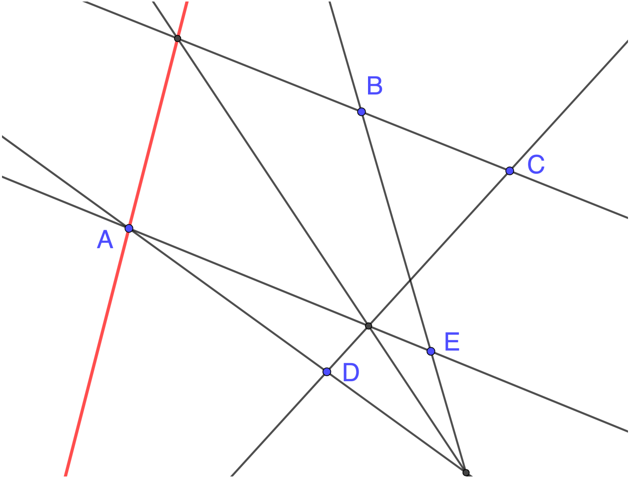
$\langle\ [a\ \langle [b\ c]\ [d\ e]\rangle]\ [b\ \langle [a\ d]\ [c\ e]\rangle]\ \rangle = \langle L\ M\rangle$
Matches its image under (a, b, e):
$\langle\ [b\ \langle [e\ c]\ [d\ a]\rangle]\ [e\ \langle [b\ d]\ [c\ a]\rangle]\ \rangle = \langle M\ N\rangle$
L, M and N are concurrent.



Five special lines

Generation 6 (Point types: 25, Line types: 792)

5: $[\ a\ \ \langle[bc]\ [\langle[cd][ea]\rangle\langle[be][da]\rangle]\rangle] : |S_C| = 4,\ |S_G| = 24,\ de\ bcde$



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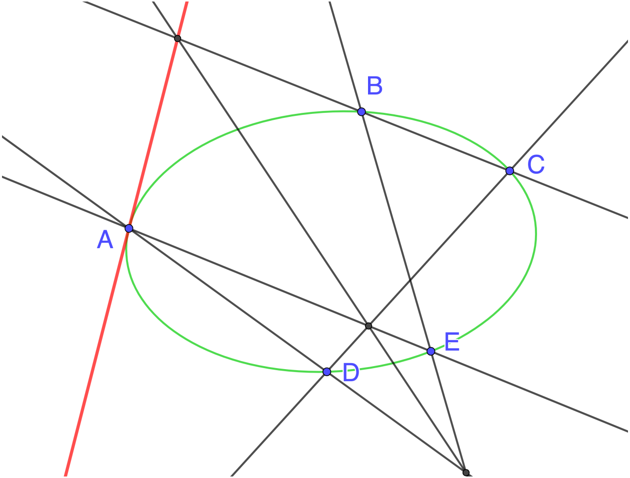

Is it a theorem?

Theorem

The recipe $[\ a\ \ \langle [bc]\ [\langle [cd][ea]\rangle \langle [be][da]\rangle]\rangle]$ *gives a ruler construction for the tangent at* a *to the conic* $abcde$.

Proof.

The hexagon $aaebcd$ is inscribed in the conic.
Its Pascal line L is $[\langle [cd][ea]\rangle \langle [be][da]\rangle]$.
L also passes through the intersection of bc and the tangent at a.
Therefore: $\langle [bc]\ [\langle [cd][ea]\rangle \langle [be][da]\rangle]\rangle$ lies on the tangent at a.

A mystery

At generation 7:

$$
\begin{array}{l}
\langle \\
\quad [a \;\; \langle[\langle[ba][cd]\rangle\langle[bc][ad]\rangle] \;\; [\langle[ba][ce]\rangle\langle[bc][ae]\rangle]\rangle] \\
\quad [\langle[bc][de]\rangle \;\; \langle[b\langle[ca][de]\rangle] \;\; [c\langle[ba][de]\rangle]\rangle] \\
\rangle
\end{array}
$$

Has combinatorial count 15 and geometric count 5.

Mystery Picture